Expressing the coefficients of the chromatic polynomial in terms of induced subgraphs: a systematic approach*

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Abstract

We follow works of Whitney, Farrell, and Morgan and Delbourgo, to express the coefficients of the chromatic polynomial $P(G;\lambda)$ of a graph G in the variable λ in terms of the number of (induced) subgraphs of G: the coefficient of $\lambda^{|G|-p}$ is given as a polynomial on variables $\binom{x_i}{k}$ with integer coefficients, and where the x_i are the number of induced copies of a 2–connected graphs with $\leq p+1$ vertices that are not formed by gluing two 2–connected graphs through a common clique. Our main contribution is that the finding of these expressions can be systematised, and that they do not depend on the 2–connected graphs with $\leq p+1$ vertices that are formed by gluing two 2–connected graphs through a common clique. As an application, we give an alternative proof of the chromatic uniqueness of the wheels with an odd number of vertices.

1 Introduction

The chromatic polynomial of a graph G, $P(G;\lambda)$, gives, as its evaluations on the positive integers n, the number of proper colourings of a graph using n colours. In particular, the chromatic polynomial has $0, 1, \ldots, \chi(G)-1$ as roots. In general, it can be defined as the polynomial that is λ on a graph on a single vertex, 0 if the graph has any loops, multiplicative over connected components, and such that $P(G;\lambda) = P(G-e;\lambda) - P(G/e;\lambda)$ when e is a non-loop edge. In general, using [6], the chromatic polynomial can be given as:

$$P(G;\lambda) = \sum_{A\subseteq E(G)} (-1)^{|A|}\lambda^{k(A)} \tag{1}$$

where $k(A)$ is the number of components of the graph $G = (V(G), A)$. In particular, the coefficient of $\lambda^{|V(G)|-p}$ is given, up to a sign, by the number of subsets of edges spanning a subgraph of G with $|V(G)|-p$ components. Whitney [6] gave the following expression for chromatic polynomial of a graph G of order n as $P(G;\lambda) = \sum_{i,j}(-1)^{i+j}m_{ij}\lambda^{n-i}$ where m_{ij} is the number of 2-connected subgraphs of G of rank i and nullity j. He [7] showed that this could be expressed as $P(G;\lambda) = \sum_i m_i\lambda^{n-i}$ where $m_i = \sum_j(-1)^{i+j}m_{ij}$ and $(-1)^i m_i$ is the number of subgraphs of G with i edges and containing no broken circuits. Building on this work, Farrell [4] showed that the coefficients of the chromatic polynomial could be expressed as

$$P(G;\lambda) = \sum_i c_{n-i}\lambda^{n-i} \tag{2}$$

where the c_{n-i} is an expression in the counts of 2-connected induced subgraphs of G (we count subgraphs and induced subgraphs in terms of edge sets of G, see (3)), however, the arguments for the general case would be quite involved.

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In the main result of this work, Theorem 2, we give a more precise description of how c_{n-i} can be written as an expression in the counts of 2-connected induced subgraphs, and also allows to easily implement an algorithm that finds such expression: the complexity of the algorithm for $n-i$ depends on the cube of the number of connected graphs on $i+1$ vertices (see Section 3 below).

Given A, a set of edges of G, the graph $(V(A), A)$ is the graph with A as its set of edges, and where $V(A) = \{v \in V(G) \mid \exists e \in A, v \text{ adjacent to } e\}$ is its set of edges. Given a graph H, we let

$$\mathrm{sube}(H,G) = \sum_{A\subseteq E(G)} \mathbf{1}_{(V(A),A) \text{ isomorphic to } H} \;\Big\|\; \mathrm{inde}(H,G) = \sum_{A\subseteq E(G)} \mathbf{1}_{G \text{ restricted to } V(A) \text{ isomorphic to } H} \quad (3)$$

be, respectively, the *number of subgraphs* isomorphic to H in G and the *number of induced subgraphs* isomorphic to H in G.[1] Regarding the previous work on some specific coefficients, the following are found in [4]:

Theorem 1. *[4, Thm 1, 2] The coefficients c_{n-3}, c_{n-4} from (2) in $P(G;\lambda)$ equals,[2], respectively*

$$-\binom{m}{3} + (m-2)t + C_4 - 2K_4 := -\binom{inde(K_2,G)}{3} + (inde(K_2,G)-2)inde(K_3,G) + inde(C_4,G) - 2inde(K_4,G) \;,$$

$$\binom{m}{4} - \binom{m-2}{2}t + \binom{t}{2} - (m-3)\cdot C_4 + (2m-9)\cdot K_4$$
$$- 6\cdot inde(K_5,G) - inde(C_5,G) + inde(\theta_{2,2,2},G) + 3inde(W_5,G) + 2inde(W_5 \setminus \{spoke\},G) \,.$$

At this point it is worth observing that Whitney's [6] main interest was to give a general account on the expressions that appear in the general coefficient of λ^{n-i} in terms of 2-connected subgraphs, while in [3, 4] the primary focus was to give an expression in terms of induced subgraphs and with the minimum number of terms as possible; the price to pay was that only the first terms could be computed (with reasonable effort) exactly. In the proof of Theorem 2, we follow the arguments of both [6, 3, 4] with the aim of giving a general account of the coefficients (in the style of [6]), but in terms of induced subgraphs (as in [3, 4]).

Remark. The fact that coefficients c_{n-i}, and, more generally, the whole chromatic polynomial of G, depends on the counts of its finite subgraphs has been extensively used in the literature, see, for instance [1, 2, 5].

2 Our result

Let $\mathcal{B}$ denote the set of 2–connected graphs. Consider the multiset of elements of $\mathcal{B}$, $\mathcal{T} = \{T_1, ..., T_1, ..., T_r, ..., T_r\}$, with t_i copies of T_i. Then $\Gamma(\mathcal{T}) = (T_1, \ldots, T_r)$ provides a sequence of elements of $\mathcal{T}$ without repetition, $n(\mathcal{T}) = (t_1, \ldots, t_r)$ and $v(\mathcal{T}) = (|V(T_1)|, \ldots, |V(T_r)|)$ give, respectively, the sequence of the number of copies that each T_i has in $\mathcal{T}$ and the number of vertices of each graph in $\mathcal{T}$ (these two sequences have an ordering consistent with $\Gamma(\mathcal{T})$). Note that a multiset of graphs, such as $\mathcal{T}$ can be viewed as a graph, denoted as $G(\mathcal{T})$, with vertex set $\sqcup_{T\in\mathcal{T}} V(T)$ and edge set $\sqcup_{T\in\mathcal{T}} E(T)$, thus having $t_1 + \ldots + t_r$ connected components.

A 2-connected graph $G = (V,E)$ is said to be *clique-separable* if there is a partition of V into three non-empty vertex sets $V = V_1 \sqcup V_2 \sqcup V_3$ such that there are no edges between V_1 and V_3, V_2 is a complete graph on $|V_2| \geq 2$ vertices, $V_1 \sqcup V_2$ and $V_3 \sqcup V_2$ induce two 2–connected graphs with $\geq |V_2| + 1$ vertices each.

[1] The number of subgraphs (induced subgraphs) usually refers to the number of injective graph homomorphisms (injective and also preserving non-edges). We are considering the subgraphs as subsets of edges; thus the number of subgraphs of C_4 in C_4 is 1, while the number of subgraphs of C_4 in C_4 with the usual understanding is 8.

[2] Note there is a typographical error in [4, Thm 2], namely "$-\binom{t}{2}$" should actually be "$+\binom{t}{2}$".

Theorem 2. *The chromatic polynomial $P(G;\lambda)$ can be computed as*

$$P(G;\lambda) = \sum_{p=0}^{|G|} \left[\sum_{\substack{\mathcal{T} \text{ multiset of } \mathcal{B} \\ (v(\mathcal{T})-(1,\dots,1))\cdot n(\mathcal{T}) \le p}} c_p(\mathcal{T}) \prod_{i\in[dim(n(\mathcal{T}))]} \binom{inde(\Gamma(\mathcal{T})_i, G)}{n(\mathcal{T})_i} \right] \lambda^{|G|-p} \tag{4}$$

$$P(G;\lambda) = \sum_{p=0}^{|G|} \left[\sum_{\substack{\mathcal{T} \text{ multiset of } \mathcal{B} \\ (v(\mathcal{T})-(1,\dots,1))\cdot n(\mathcal{T}) \le p}} s_p(\mathcal{T}) \prod_{i\in[dim(n(\mathcal{T}))]} \binom{sube(\Gamma(\mathcal{T})_i, G)}{n(\mathcal{T})_i} \right] \lambda^{|G|-p} \tag{5}$$

where: $(v(\mathcal{T}) - (1,\dots,1)) \cdot n(\mathcal{T})$ is the usual scalar product of two vectors, $inde(\cdot, G)$ and $sube(\cdot, G)$ are given by (3), both $c_p(\mathcal{T})$ and $s_p(\mathcal{T})$ are integers depending solely on $\mathcal{T}$ and p (not on G), and $|G| := |V(G)|$. Furthermore:

(i) $c_p(\mathcal{T}) = 0$ *if* $(v(\mathcal{T}) - (1,\dots,1)) \cdot n(\mathcal{T}) > p$

(ii) $c_p(\mathcal{T}) = 0$ *if a* $T \in \mathcal{T}$ *is clique-separable*

(iii) if $\mathcal{T} = \{T\}$ *and* $|T| = p+1$,

$$c_p(\mathcal{T}) = \sum_{A\subseteq E(T),\ (V(T),A)\ 2\text{-}connected} (-1)^{|A|}$$

(iv) if $\mathcal{T} = \{T\}$ *and* $|T| = p+1$ *and* $i \ge 1$,

$$c_{p+i}(\mathcal{T}) = - \sum_{\substack{\mathcal{T}' \text{ multiset of } \mathcal{B},\ \mathcal{T}' \neq \mathcal{T} \\ \mathcal{T}' \text{ containing subgraphs of } T}} c_{p+i}(\mathcal{T}') \prod_{j\in[dim(n(\mathcal{T}'))]} \binom{inde(\Gamma(\mathcal{T}')_j, T)}{n(\mathcal{T}')_j}$$

(v) When $|\mathcal{T}| = t \ge 2$ *and for each* $i \ge 0$ *we have:*

$$c_{(v(\mathcal{T})-(1,\dots,1))\cdot n(\mathcal{T})+i}(\mathcal{T}) = \sum_{k_1+\ldots+k_t=i,\ k_s\ge 0} \ \prod_{T_t\in\mathcal{T}} c_{|T_t|-1+k_t}(\{T_t\})\ .$$

(vi) For each $p \ge 0$ *and* $\mathcal{T}$ *multiset of* $\mathcal{B}$, $c_p(\mathcal{T})$ *are determined by (i),(ii),(iii), (iv), (v).*

Before proceeding to the proof, we highlight that, in the proof and in the statement of Theorem 2, the H in (3) that are used are 2–connected.

Sketch of the proof. First we show (5) by translating the summation over edges as a sum of "independent" combinations of 2–connected blocks which would form the subgraph in question. Since we are considering these 2–connected blocks as being combined independently, we should substract the instances where these independent 2–connected blocks are combined into larger 2–connected blocks, such as when 3 edges are combined to form a triangle. Once (5) is obtained, we show (4) using that the number of instances of a subgraph T can be counted using induced graphs that are supergraphs of T on the same vertex set. We complete the argument using Vandermonde's involution formula together with Pólya and Ostrowski result from 1920 which implies that the polynomials $\binom{mx}{k}$ with positive integers m and k and variable x can be written in terms of $\binom{x}{i}$, $1 \le i \le k$ using integer coefficients. Part (iii) follows by examining the contribution to c_{n-i} in (2) by only one 2–connected block with $i+1$ vertices. Parts (ii), (v), and (iv) follow by the multiplicative properties of the chromatic polynomial over 2–connected components, and its behaviour over clique-join graphs. In particular, given (4) and the $c_p(\mathcal{T})$ as unknowns we consider certain chromatic polynomials which, when closely examined, gives the equations and relations described in (ii), (v), and (iv). These constructions are described below.

Proof of (ii). Let A be a 2-connected graph which is a clique-join of two other graphs (so it is clique-separable). The proof goes by induction on the number of edges and vertices of A. Let A_1, A_2 be the two 2-connected components that are joined by a clique. Then we consider G_1 the graph formed by 5 vertex-disjoint copies of A and G_2 the graph obtained by the disjoint union of: 2 vertex-disjoint copies of A_1, 2 vertex-disjoint copies of A_2, and a graph formed by 3 copies of A_1 and 3 copies of A_2 on the same clique (and in such a way that the number of induced copies of A in the resulting graph is 9 by choosing one of the copies of A_1 and one of the copies of A_2, independently). The chromatic polynomial of G_1 and G_2 is the same in both cases:

$$\left(\frac{P(A_1;\lambda)P(A_2;\lambda)}{\lambda(\lambda-1)\cdots(\lambda-q)}\right)^5 = \frac{\frac{\left(\frac{P(A_1;\lambda)P(A_2;\lambda)}{\lambda(\lambda-1)\cdots(\lambda-q)}\right)\left(\frac{P(A_1;\lambda)P(A_2;\lambda)}{\lambda(\lambda-1)\cdots(\lambda-q)}\right)}{\lambda(\lambda-1)\cdots(\lambda-q)}\left(\frac{P(A_1;\lambda)P(A_2;\lambda)}{\lambda(\lambda-1)\cdots(\lambda-q)}\right)}{\lambda(\lambda-1)\cdots(\lambda-q)} P(A_1;\lambda)^2 P(A_2;\lambda)^2$$

where $q+1$ is the size of the clique by which they are joined in A. Moreover:

- If a 2-connected graph T is an induced graph of one of the parts (either a subgraph of A_1 or a subgraph of A_2), then the number of induced copies of T in G_1 and in G_2 (counted as in (3)) is the same.
- There are strictly more induced copies of A in G_2 than in the former (5 in G_1 versus 9 in G_2 if we assume A_1 and A_2 are different, otherwise is 5 versus at least 9 or at most $\binom{6}{2} = 15$ depending on the join interaction of A_1 and A_2 with respect to the common clique).
- Any induced copy of a 2–connected graph that contains a part in A_1 and a part of A_2 (and a strict induced subgraph of A) would consists on two graphs joined through a clique, and thus the corresponding coefficient in any chromatic polynomial and for any monomial is zero by induction.

By the previous argument for any induced graph completely contained in A_1 or A_2 the induced graph accounts are the same, and any graph that contains a part in A_1 and a part in A_2 is a clique-join and thus, by an inductive argument, does not appear in the summation making up the coefficients. Therefore, the fact that there are strictly more induced copies of A in G_2 than in G_1 implies that the corresponding coefficient of A for λ in $P(A;\lambda)$ should be zero. By adding some isolated vertices, we can conclude the same for all the coefficients involving the number of induced copies of A, when $\mathcal{T} = \{A,\dots,A\}$, are zero. For a general multiset $\mathcal{T}$ containing A, it follows from (v) and the fact that all the coefficients are zero when $\mathcal{T} = \{A\}$ as we have just shown.

Proof of (iv). Consider the chromatic polynomial of T, a 2–connected graph and $i \geq 1$ isolated vertices; $T \sqcup \{v_j\}_{j\in[i]}$. Then, $P(T \sqcup \{v_j\}_{j\in[i]};\lambda) = P(T;\lambda)\lambda^i$. From (4), we can determine that there are no 2–connected components with larger number of vertices (or edges) than T, so all the terms $\binom{\mathrm{inde}(\Gamma(\mathcal{T})_i, T\sqcup\{v_j\}_{j\in[i]})}{n(\mathcal{T})_i}$ are zero unless $\Gamma(\mathcal{T})_i$ is an induced subgraph of T. In particular, the only coefficients $c_p(\mathcal{T})$ that are multiplying non-zero terms of the type $\prod_{j\in[\dim(n(\mathcal{T}))]}\binom{\mathrm{inde}(\Gamma(\mathcal{T})_j, G)}{n(\mathcal{T})_j}$ are those where all the graphs in $\mathcal{T}$ are induced subgraphs of T. This implies that, if $p = |T|-1+i$ we have

$$0 = \left[\sum_{\substack{\mathcal{T} \text{ multiset of } \mathcal{B},\ (v(\mathcal{T})-(1,\dots,1))\cdot n(\mathcal{T}) \leq |T|-1+i \\ \mathcal{T} \text{ containing only induced subgraphs of } T}} c_{|T|-1+i}(\mathcal{T}) \prod_{j\in[\dim(n(\mathcal{T}))]} \binom{\mathrm{inde}(\Gamma(\mathcal{T})_j, G)}{n(\mathcal{T})_j}\right] \tag{6}$$

where the zero comes from the fact that all the coefficients multiplying monomials from $P(T \sqcup \{v_j\}_{j\in[i]};\lambda)$ of degree $< i+1$ are zero, which implies that when $p = |T|-1+i$ the coefficient of $\lambda^{|T|+i-|T|+1-i} = 0$. Isolating the term $c_{|T|-1+i}(T)$ in (6) that is multiplying $\binom{\mathrm{inde}(T,\,G)}{1} = 1$ gives (iv).

Proof of (v). We have to show that, for each $i \geq 0$, $c_{(v(\mathcal{T})-(1,\dots,1))\cdot n(\mathcal{T})+i}(\mathcal{T}) = \sum_{k_1+\ldots+k_t=i, k_s\geq 0, \text{ with } |\mathcal{T}|=t} \prod_{T_j\in\mathcal{T}} c_{|T_j|-1+k_j}(\{T_j\})$. Given $\mathcal{T} = \{T_1,\dots,T_1,T_2,\dots,T_2,\dots,T_r,\dots,T_r\}$ with t_i copies of each T_i, consider the chromatic polynomial of G, the graph obtained from the disjoint union of t_j copies of the graph T_j, for each $j \in [r]$, and i isolated vertices, so: $P(G;\lambda) = P(\sqcup_{j\in[r], s\in[t_i]} T_j \sqcup \{v\}_{j\in[i]};\lambda) =$

$\lambda^i \prod_{j\in[r]} P(T_j;\lambda)^{t_j}$. By the disjoint unionness of G (in terms of the graphs of $\mathcal{T}$), for each $T_j \in \mathcal{B}$, $\mathrm{inde}(T_j, G) = \sum_{T\in\mathcal{T}} \mathrm{inde}(T_j, T)$. Using Vandermonde's involution formula to the latter, the monomial involving the coefficient $c_{(v(\mathcal{T})-(1,\dots,1))\cdot n(\mathcal{T})+i}(\mathcal{T})$ which multiplies the term $\prod_{j\in[\dim(n(\mathcal{T}))]} \binom{\mathrm{inde}(\Gamma(\mathcal{T})_j,G)}{n(\mathcal{T})_j} = \prod_{j\in[\dim(n(\mathcal{T}))]} \binom{\mathrm{inde}(T_j,G)}{t_j}$ depends on the multiplication of the terms where$\binom{\mathrm{inde}(T_i,T_i)}{1}$ from the polynomial $P(T_i;\lambda)$,[3] while making that the powers of the λ coincide at the end. Thus the formula (v) follows.

3 Comments and consequences of our main result

On the $c_p(\mathcal{T})$ of chromatically equivalent graphs. Observing Theorem 2, it is natural to ask the relationship between pairs of graphs G and H, their chromatic polynomials $P(G;\lambda)$ and $P(H;\lambda)$, and the relationship between their corresponding sequences $\{c_p(G)\}_p$ and $\{c_p(H)\}_p$. Even though there is obviously a relation, it is non-trivial. In particular, there are pairs of graphs with the same sequences $\{c_p(G)\}_p = \{c_p(H)\}_p$, yet $P(G;\lambda) \neq P(H;\lambda)$, for instance, any pair of graphs that are clique-joins, yet they have different chromatic polynomials (even perhaps they have a different number of vertices). On the other hand, the following two graphs, shown as Figure 1 have the same chromatic polynomial:

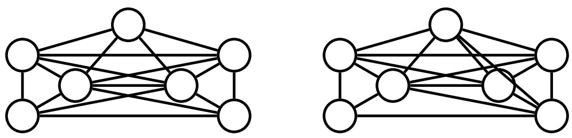

Figure 1: Graph G on the left. Graph H on the right

$P(G;\lambda) = P(H;\lambda) = \lambda^7 - 17\lambda^6 + 118\lambda^5 - 425\lambda^4 + 829\lambda^3 - 818\lambda^2 + 312\lambda$, yet their sequences $\{c_p(G)\}_p$ differ. Indeed, G from Figure 1 is clique-separable (vertices $V_1 = \{\text{top vertex}\}, V_2 = \{\text{middle four}\}, V_3 = \{\text{lower two}\}$, so all its coefficients are 0, however, H is not clique-separable, and in particular, it has a non-zero coefficient $c_6(H) = 30$.

Algorithmic questions. Expression (4) in Theorem 2 can be used in order to find $c_{n-p}(\mathcal{T})$ as follows. One can set a linear system with one equation for each connected graph G on $p+1$ vertices using the value of the coefficient of λ^{n-p} in the expression for $P(G;\lambda)$, with the unknown coefficients $c_{n-p}(\mathcal{T})$ in the expression (4) as variables, and with the corresponding expression of the number of induced subgraphs as coefficients of the equation (for each graph G, these numbers can be computed). The number of connected graphs on $p+1$ vertices is denoted by k_p. k_p is then an upper bound on the number of variables, and on the number of equations as well. Thus the linear system can be solved in time $O(k_p^2)$. Also, for each graph G, its linear equation can be set up in time $2^{|E(G)|}$ times checking the 2–connected isomorphism type of the subset of edges; since $2^{|E(G)|} = O(k_p)$ and checking the 2–connected isomorphism has $O(p!k_p) = O(k_p^2)$ complexity, a relatively easy algorithm on time $O(k_p^3)$ can be implemented.

On wheels. As W_{2n-1} are 3-colourable, no induced copy of K_4 can be found in a graph chromatically equivalent to them. Then Lemma 3 gives an alternative proof that W_{2n-1} are chromatically unique [8].

Lemma 3. *If G is a graph on $n \geq 5$ vertices and chromatically equivalent to the wheel W_n (that is, $P(G;\lambda) = P(W_n;\lambda)$), and $G \ncong W_n$, then G has at least 2 induced C_4 and an induced K_4.*

[3]The lower term of the binomial coefficient cannot be strictly larger, as there are no sufficient copies in T_i. If it is strictly smaller, then will be accounted by some $\binom{\mathrm{inde}(T_j,T_j)}{i}$ where i is strictly larger. Further, for $\binom{\mathrm{inde}(T_i,T_j)}{t}$ with $i \neq j$, the only way of carrying such term is when the product picks another $\binom{\mathrm{inde}(T_a,T_b)}{t}$, with $i > 0$ and where T_a is not a subgraph of T_b, and thus the term will become 0.

Proof. G and W_n should have the same number of vertices. Since the chromatic coefficients of λ^{n-1}, and λ^{n-2} are the same for G and W_n, they have the same number of edges and triangles.

Since any induced 2–connected subgraph of the wheel with n vertices (maximal induced cycle of size $n-1$) with ≥ 4 and $\leq n-2$ vertices is a clique-join, we may use Theorem 2 (ii) and (4), the expressions configuring the coefficients of λ^{n-p} for $p \in [3, n-3]$ for the wheel only depends on the number of triangles and number of edges, and thus these expressions gets balanced with those parts from G (as they have the same number of triangles and edges).

Now, since the chromatic number of the wheels is 4, there are no induced copies of K_j, $j \geq 5$ in G. By Theorem 2 or Theorem 1, the graph G will have induced K_4 if and only if it has induced C_4's; indeed, when $n \geq 6$ is due to Theorem 1 and the fact that W_n has neither induced C_4 nor K_4, and thus these numbers should balance in the coefficient λ^{n-3} as the coefficient also depends on the number of triangles and edges, but those two numbers are the same for G and for W_n (when $n = 5$, G could have a C_4 without a K_4 but then it would be the wheel as those two have the same number of triangles and triangles all should be incident with the last vertex, and thus the claim follows).

Assume for a contradiction that G has no induced $C_4, C_5, \ldots, C_k$, for some $k \geq 4$, then any induced 2–connected graph on $\leq k$ vertices has only triangles, and all the induced cycles have a chord. In particular, all of these graphs are chordal graphs. Thus, it has a perfect elimination ordering, meaning that the neighbourhood of the any removed vertex in the perfect elimination ordering is a clique. This means that, either the graph is K_i, $i \geq 4$, or it is a clique-join. Since G, $n \geq 5$ has no copies of K_i, $i \geq 5$, the only remaining case is for K_4. However, if it has a K_4, then G should also contain an induced C_4 (as claimed in the previous paragraph). Now let us focus on the coefficient c_{n-k}. Consider an induced 2-connected subgraph of G with $k+1$ vertices; if it is not a chordal graph and it is not C_{k+1}, then it contains an induced cycle on $\leq k$ vertices, which is a contradiction with the assumption. Otherwise, it is a chordal graph, and the perfect elimination ordering shows that it is either a clique on $k+1$ vertices, or a clique-join, thus not counting towards c_{n-k}. In particular, it can only be C_{k+1}, but if $n > k+2$, then W_n has no induced cycle of length C_{k+1} and the only contributions towards c_{n-k} are from edges and triangles, which is the same as for G. Since C_{k+1} has a non-zero coefficient by Theorem 2 (iii), then G cannot have an induced copy of C_{k+1} for otherwise, under the assumption of having no $C_4, C_5, \ldots, C_k$, then it would not have the same chromatic polynomial as W_n. This process can be run until c_2 for which a single copy of C_{n-1} appears in W_n, thus by the previous argument forcing a single copy of C_{n-1} in G as well. Since both have the same number of triangles ($n-1$ induced triangles and $2(n-1)$ edges is the wheel on n vertices), they end up being the same graph, and thus the assumption $G \neq W_n$ does not hold. □

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