Three-term arithmetic progressions in two-colorings of the plane*

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Abstract

We show that in any two-coloring of the plane, there exists a monochromatic congruent copy of any arithmetic progression of length 3. This problem lies at the intersection of two longstanding but active research projects. The first is the study of Ramsey problems for arithmetic progressions in colorings of euclidean space, for which there are many results dating back over 50 years, but about which much is still not known. The second is centered around a conjecture of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus, which posits that any two-coloring of the plane must contain a monochromatic congruent copy of every non-equilateral three-point configuration. Our result confirms one of the most natural open cases of this conjecture.

1 Introduction

We let $\mathbb{E}^n$ denote n-dimensional Euclidean space, that is, $\mathbb{E}^n$ equipped with the Euclidean norm. The field of Euclidean Ramsey theory is concerned with what types of configurations (monochromatic, rainbow, etc.) must exist in *any* coloring of $\mathbb{E}^n$ using a prescribed number of colors. One of the most commonly studied configurations is denoted ℓ_m, and consists of m collinear points with consecutive points at distance 1 apart. In other words, ℓ_m is an m-term arithmetic progression with common difference 1. Our main result is the following.

Theorem 1. *In any two-coloring of* $\mathbb{E}^2$*, there exists a monochromatic congruent copy of* ℓ_3*.*

Thus, by scaling, there naturally exists a monochromatic 3-term arithmetic progression with any common difference. The classical question in this area, known as the Hadwiger-Nelson (HN) problem, is one of the most famous open problems in combinatorics. The HN problem, first discussed by Nelson (not in print) in 1950, asks how many colors one would need to color $\mathbb{E}^2$ so that there is no monochromatic copy of ℓ_2; i.e. two points of unit distance apart. This quantity is known sometimes as the *chromatic number* $\chi(\mathbb{E}^2)$ of the plane. It was known that the answer is between 4 and 7 for a long time, and a 2018 breakthrough by de Grey [7] showed that one needs at least 5 colors. In general, it is known that $(1.239 + o(1))^n \leq \chi(\mathbb{E}^n) \leq (3 + o(1))^n$ as $n \to \infty$ [12, Section 11.1].

After the introduction of the HN problem, the area was further developed by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus in a series of papers [8, 9, 10]. In these papers, they ask if, for any non-equilateral three-point configuration K, there must be a monochromatic congruent copy of K in any 2-coloring of $\mathbb{E}^2$. The conjecture was confirmed when the coloring is assumed to be polygonal [14], but it is still widely open in general. As noted in [2, Section 6.3], Theorem 1 gives perhaps

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the most natural open case of this conjecture. This problem was discussed as well in the concluding remarks of a very recent paper of Führer and Tóth [11, Page 12].

To discuss further known results, we introduce some standard notation. If we have configurations $K_1, \ldots, K_r$ in $\mathbb{E}^n$, we say that $\mathbb{E}^n \to (K_1, \ldots, K_r)$ if, for any coloring of $\mathbb{E}^n$ with r colors, there exists a monochromatic (congruent) copy of K_i in color i, for some i. If there exists a coloring where this does not hold, we say $\mathbb{E}^n \not\to (K_1, \ldots, K_r)$. For simplicity, if $K_i = K$ for all i and $\mathbb{E}^n \to (K_1, \ldots, K_r)$ or $\mathbb{E}^n \not\to (K_1, \ldots, K_r)$, we say simply $\mathbb{E}^n \xrightarrow{r} K$ (resp. $\mathbb{E}^n \overset{r}{\not\to} K$).

Using the above terminology, our Theorem 1 says that $\mathbb{E}^2 \xrightarrow{2} \ell_3$. The question of for which $n, r, s_1, \ldots, s_r$ we have $\mathbb{E}^n \to (\ell_{s_1}, \ldots, \ell_{s_r})$ also has a rich history, so we collect here the known results. Perhaps the most relevant results to this manuscript are that $\mathbb{E}^2 \overset{3}{\not\to} \ell_3$, that $\mathbb{E}^3 \xrightarrow{2} \ell_3$, and that there exists m such that $\mathbb{E}^n \not\to (\ell_3, \ell_m)$ for all n. The first of these results was shown by Graham and Tressler [13] using a simple hexagonal grid construction. In [8, Theorem 8] Erdős et. al. proved that $\mathbb{E}^3 \xrightarrow{2} T$ for any triangle[1] T; in particular, the second result, that $\mathbb{E}^3 \xrightarrow{2} \ell_3$. The third result, that there exists m such that $\mathbb{E}^n \not\to (\ell_3, \ell_m)$ for all n, is a recent result of Conlon and Wu [4]. They were able to show a bound of $m = 10^{50}$, and in a recent paper, Führer and Tóth [11] were able to improve this to $m = 1177$. Some other relevant results in the area are as follows.

- $\mathbb{E}^2 \to (\ell_2, K)$ for any K with 4 points (Juhász [15])
- $\mathbb{E}^2 \to (\ell_2, \ell_5)$ (Tsaturian [17])
- There is a set K with 8 points, such that $\mathbb{E}^2 \not\to (\ell_2, K)$ (Csizmadia and Tóth [5])
- $\mathbb{E}^3 \to (\ell_2, \ell_6)$ (Arman and Tsaturian [1])
- $\mathbb{E}^n \not\to (\ell_2, \ell_{2^{cn}})$ for some constant $c > 0$ (Conlon and Fox [3])
- $\mathbb{E}^n \overset{2}{\not\to} \ell_6$ (Erdős et. al. [8, Theorem 12])

An (a, b, c) *triangle* is a triangle with side lengths a, b, c. The following theorem is due to Erdős et. al. [10, Theorem 1].

Theorem 2 (Erdős et. al.). *A given 2-coloring admits a monochromatic (a, b, c) triangle if and only if it admits a monochromatic equilateral triangle of side a, b, or c.*

Note that ℓ_3 is a $(1, 1, 2)$ triangle. Thus, by scaling, Theorem 1 and Theorem 2 imply the following corollary.

Corollary 3. *If $n \geq 2$, then $\mathbb{E}^n \xrightarrow{2} T$ for an $(\alpha, 2\alpha, x\alpha)$ triangle T for any $\alpha > 0$ and $x \in [1, 3]$.*

This verifies another interesting case of the aforementioned conjecture of Erdős et. al. from [10]. We refer to [10, 16] and [12, Theorem 11.1.4 (a)] for a collection of known families of triangles T such that $\mathbb{E}^2 \xrightarrow{2} T$. In particular, Erdős et. al. [10] showed that $\mathbb{E}^2 \xrightarrow{2} T$ if T has a ratio between two sides equal to $2\sin(\theta/2)$ with $\theta \in \{30°, 72°, 90°, 120°\}$. Our result handles the case that $\theta = 180°$.

In the next section, we will give an outline of the proof of Theorem 1.

2 Sketch of proof

In this section, we will discuss the main ideas of the proof of Theorem 1. We start with a simple corollary of Theorem 2.

[1]Throughout, degenerate triangles (that is, three collinear points) are also regarded as triangles.

Corollary 4. *If a coloring of $\mathbb{E}^2$ does not contain a monochromatic ℓ_3, then it also does not contain a monochromatic equilateral triangle of side-length 1 or 2.*

Our proof of Theorem 1 will proceed in two parts, both with the same general outline. Each will proceed by contradiction, starting with the assumption that there exists a coloring of $\mathbb{E}^2$ that has no monochromatic ℓ_3. Then, we begin with a small set of starting points, and show that all possible colorings of those starting points will result another point that must be colored both blue and red; that is, a contradiction. So far, we have the following "rules" at our disposal that will allow us to execute this proof: we can take two points of the same color at distance 1 or 2 apart, and do one of the following.

- Add a third point of the opposite color to form an ℓ_3 (as a midpoint if the points are distance 2 apart), or
- add a third point of the opposite color to create an equilateral triangle (of side-length 1 or 2).

Where the second option follows directly from corollary 4. Visually, we can think of these rules as follows.

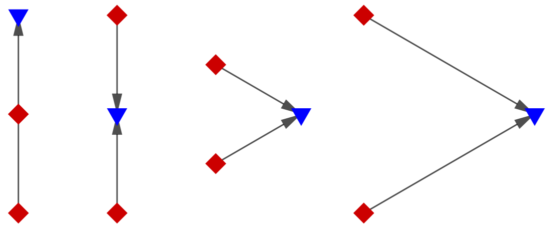

Figure 1: The color implication steps

These two rules are not quite enough to establish Theorem 1. Thus, the first part of the proof, detailed in Section 2.1, will be to establish one more useful rule. Then, in section 2.2 we will describe how to use our rules to prove Theorem 1.

2.1 Another rule

The goal of this section is to describe the following result.

Lemma 5. *In any two-coloring of $\mathbb{E}^2$ containing no monochromatic ℓ_3, any unit triangle colored blue-blue-red has a blue centroid.*

By a symmetric argument, under the same assumptions any red-red-blue triangle has a red centroid. To outline the proof of this result, we need an efficient way to describe the coordinates of our point sets. If a, b, c, d are integers, then all points we use will be of the following form:

$$[a, b, c, d] := \left(\frac{a\sqrt{3} + b\sqrt{11}}{12}, \frac{c + d\sqrt{33}}{12} \right). \tag{1}$$

The proof will proceed as follows: we will start with a basic pointset, containing a unit equilateral triangle and its centroid. Then, we will assume the triangle is colored blue-blue-red but has a red centroid, and use the rules from the previous section (as in Figure 1) to derive a contradiction. The pointset we will use is drawn in Figure 2, with the following explicit coordinates.

$$\begin{aligned} p_1 &= [-4, 0, 0, 0], & p_2 &= [0, 0, 0, 0], & p_3 &= [2, 0, -6, 0], & p_4 &= [2, 0, 6, 0], \\ q_1 &= [-1, -3, 3, -1], & q_2 &= [-1, -3, -3, 1], & q_3 &= [2, 0, 0, 2], & q_3' &= [2, 0, 0, -2], \\ q_4 &= [-3, -3, -3, -1], & q_5 &= [-3, -3, 3, 1]. \end{aligned}$$

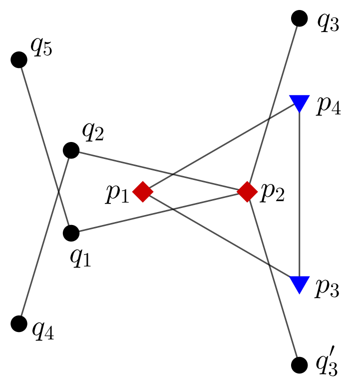

Figure 2: The base points needed to verify the lemma

We will need to consider all possible colorings of this pointset, which results in some case work. However, the symmetries present allow us to limit this to only 6 cases, and the color implications we end up with are simple enough to be verified (somewhat tediously) by hand. Alternatively, we provide a method to quickly verify the result computationally. We again refer to [6] for a complete description of these results, but for the moment we use Figure 3 to visualize the simplest case - that is, where q_1 and q_2 have different colors.

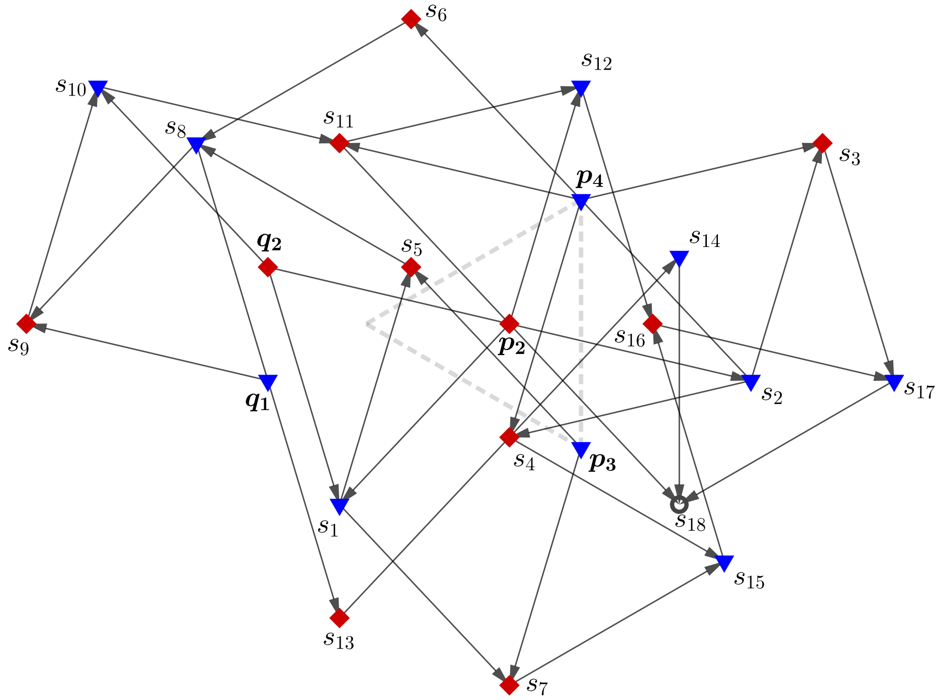

Figure 3: Case 1: q_1 is red and q_2 is blue (or vice versa)

The contradiction comes from the fact that the point s_{18} must be colored both red and blue; the blue coloring comes from the ℓ_3 created with red points s_{11} and p_2, and the red coloring comes from the equilateral triangle created with blue points s_{14} and s_{17}. We note, finally, that not all points from Figure 2 are used in this case. However, the remaining cases will make use of all of the p_i and q_j.

2.2 Putting it all together

Using the rules described in section 2 as well as the one established in section 2.1, we can now complete the proof of Theorem 1. We'll deal with a $\frac{1}{\sqrt{3}}$-scaled hexagonal grid - that is, where the smallest triangles are scaled to have edge-length $\frac{1}{\sqrt{3}}$. A straightforward argument (detailed in [6]) using these rules shows that if we assume there is no monochromatic ℓ_3, then there is only one coloring of this grid up to isometry - that is, the one pictured in Figure 4.

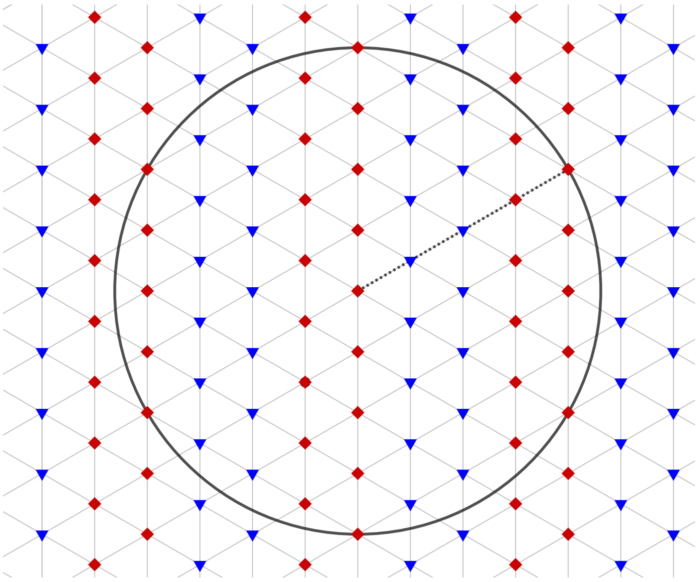

Figure 4: A circle with radius $\frac{4}{\sqrt{3}}$ in the colored grid

To finish the proof is straightforward. We pick two points in our coloring that are less than distance $\frac{8}{\sqrt{3}}$ from one another and have different colors; call them p_1 and p_2, and let them be red and blue respectively. As illustrated in Figure 4, any point on the hexagonal grid at distance $\frac{4}{\sqrt{3}}$ from p_1 must be red as well. By rotating the grid, we can actually show that all points at distance $\frac{4}{\sqrt{3}}$ from p_1 are red, and symmetrically all points at distance $\frac{4}{\sqrt{3}}$ from p_2 are blue. However, since p_1 and p_2 are of distance less than $\frac{8}{\sqrt{3}}$ from one another there must be a point that is distance $\frac{4}{\sqrt{3}}$ from both of these points, which provides our contradiction.

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