Random lifts of very high girth and their applications to frozen colourings*

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Abstract

We study the cycle distribution of a random n-lift of a fixed d-regular graph on m vertices, deriving an asymptotic formula for the probability that it has girth at least $g = g(n)$, provided that $g(n)$ grows sufficiently slowly with respect to m, d and n. As a consequence of the existence of lifts with high girth, we construct graphs with very large girth that admit frozen colourings, and graphs with moderately large girth where typical colourings are partially-frozen. The latter result shows the tightness on the girth condition of a recent theorem on graph colouring rigidity by Hurley and Pirot [STOC, 2023].

1 Introduction

An *n-lift* of a graph G is a graph $L = L_n(G)$ with vertex set $V(L) := V(G) \times [n]$ and edge set obtained as follows: for every edge $uv \in E(G)$, we place a perfect matching between the sets $\{u\} \times [n]$ and $\{v\} \times [n]$. Let $\mathcal{L}_n(G)$ be the set of all n-lifts of a graph G.

A *random n-lift of G*, denoted by $\mathbb{L}_n(G)$, is an n-lift of G chosen uniformly at random from $\mathcal{L}_n(G)$. It is worth noticing that we may generate $\mathbb{L}_n(G)$ by choosing each perfect matching corresponding to an edge in G, independently and uniformly at random.

Random lifts of graphs were introduced by Amit and Linial in 2002 [2, 3] and since then, they have attracted a lot of interest in the area. Among other works, we highlight the results of Amit, Linial and Matoušek [4] on their independent and chromatic numbers, the results of Greenhill, Janson and Ruciński [10] on their number of perfect matchings, and the work of Bordenave [6] on their spectral properties.

Fortin and Rudinsky [8] studied the distribution of short cycles in random lifts. Given a subgraph $H \subseteq L$, its *pattern* is the multigraph on $V(G)$ obtained by adding an edge (u, v) for every edge $(u, x)(v, y) \in E(H)$. The following observation is key to study the cycles of random lifts

If C is a k-cycle of $L \in \mathcal{L}_n(G)$, then the pattern of C is a closed non-backtracking k-walk in G.

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Let $w_k(G)$ be the number of closed non-backtracking k-walks in G. For all $k \geq 3$, we let

$$\lambda_k(G) := \frac{w_k(G)}{2k}.$$

and

$$\mu_k(G) := \sum_{\ell=3}^{k-1} \lambda_\ell(G).$$

Theorem 1 ([8])**.** *Let $n \in \mathbb{N}$ and $d \geq 3$, and let G be a d-regular graph. For any $k \geq 3$, let $Z_{k,n}$ be the number of cycles of length k in $\mathbb{L}_n(G)$. Let Z_k be independent random variables with Poisson distribution of parameter $\lambda_k(G)$ respectively. Then $(Z_{k,n})_{k\geq 3} \to (Z_k)_{k\geq 3}$ in distribution as $n \to \infty$.*

For any graph H, let $g(H)$ be its *girth*; the length of a shortest cycle. The previous result implies that, for every $g_0 \geq 3$,

$$\lim_{n\to\infty} \mathbb{P}(g(\mathbb{L}_n(G)) \geq g_0) = e^{-\mu_{g_0}(G)} > 0. \tag{1}$$

The qualitative behaviour of short cycles in random lifts is the same as for other random graph models, such as Erdős-Rényi random graphs or random regular graphs (see e.g. [9]). In the case of random regular graphs, McKay, Wormald and Wysocka [12] went a step further and studied the distribution of long cycles. As a corollary, they obtained an enumeration formula for d-regular graphs on n vertices with girth at least g, provided that $(d-1)^{2g-3} = o(n)$.

Our main result extends (1) in the line of [12], allowing for the girth $g(n)$ to tend to infinity when $n \to \infty$, provided it does it sufficiently slowly with respect to the other parameters.

Theorem 2. *Let $n \in \mathbb{N}$, $d = d(n) \geq 3$, $m = m(n)$ and $g = g(n)$ such that $m(d-1)^{2g-4} = o(n)$. If G is a d-regular graph on m vertices, then,*

$$\mathbb{P}(g(\mathbb{L}_n(G)) \geq g(n)) \sim e^{-\mu_{g(n)}(G)}. \tag{2}$$

An immediate corollary of our main theorem is the existence of lifts of any fixed regular graph G with very high girth.

Corollary 3. *Let $n \in \mathbb{N}$, $d = d(n) \geq 3$, $m = m(n)$ and $g = g(n)$ such that $m(d-1)^{2g-4} = o(n)$. If G is a d-regular graph on m vertices, then, for any sufficiently large n, there exists $L \in \mathcal{L}_n(G)$ with $g(L) \geq g(n)$.*

The condition on the parameters is not far from optimal. Recall Moore's bound for odd girth: the number of vertices of any d-regular graph with girth at least $g = 2s + 1$ is

$$n \geq 1 + d\sum_{i=1}^{s-1}(d-1)^i \geq (d-1)^{(g-1)/2}$$

and thus the restriction on the girth is tight up to a constant factor.

1.1 Ideas of the proof

To exemplify the ideas behind the proof of Theorem 2, we give a sketch of a direct proof of Corollary 3 that does not use the theorem. The approach is a combination of the second moment technique and the switching method, that we now detail.

One of the most important aspects is to control the expected number of appearances of k-cycles (and other subgraphs) in random lifts, which is done using the two lemmas below.

Lemma 4. *For every $k \geq 3$ satisfying $m(d-1)^{2k-4} = o(n)$, if X_k is the number of k-cycles in $\mathbb{L}_n(G)$, then*

$$\mathbb{E}(X_k) \sim \lambda_k(G).$$

A graph H is *feasible* if its vertex set is a subset of $V(G) \times [n]$ and there exists $L \in \mathcal{L}_n(G)$ with $H \subseteq L$.

Lemma 5. *Let H be a connected feasible graph on h vertices and e edges. Let Y_H be the number of subgraphs isomorphic to H in $\mathbb{L}_n(G)$. If $e = o(n^{1/2})$, then*

$$\mathbb{E}(Y_H) = O(md^{h-1}n^{h-e}).$$

We use these two results in the next lemma, which is proved by an application of the second moment method. Crucially, we also use an upper bound on the number of subgraphs H that can be obtained from the union of two cycles, that depends on the number of components and the number of edges of the intersection graph of the two cycles, that was derived in [12].

Lemma 6. *Let $s_k := \max\{2\lambda_k(G), \log^2 n\}$. Then,*

$$\mathbb{P}(X_k > s_k, \textit{ for some } 3 \leq k < g) = o(1).$$

Moreover, the probability that there are two cycles of length shorter than g that share at least one edge is $o(1)$.

With the previous lemma in hand, we give a proof of the existence of a lift of G with no cycles of length less than g (short cycles), that we now sketch.

Let L_0 be a graph that satisfies the conclusions of Lemma 6 (few short cycles and all of them edge-disjoint). A key property is that the number of vertices participating in short cycles is at most

$$\sum_{k=3}^{g-1} ks_k = o(n).$$

In the classical argument of Erdős to find graphs with large girth and large chromatic number (see e.g. [1]), an arbitrary vertex of each short cycle is deleted, which enforces the girth to be at least g. However, since we want to maintain the property that the final graph is a lift of G, we need to find an alternative way to get rid of the short cycles of L_0. The idea will be to use a switching-type argument to destroy them one by one, while keeping the structure of a lift. In doing so, we will strongly use that the cycles are edge-disjoint.

A *switch* on $L \in \mathcal{L}_n(G)$ is a local transformation defined as follows: Let $uv \in E(G)$ and $x_1, x_2, y_1, y_2 \in [n]$ such that $(u, x_1)(v, y_1)$ and $(u, x_2)(v, y_2)$ are edges of L, and $(u, x_1)(v, y_2)$ and $(u, x_2)(v, y_1)$ are not. Then, we delete the former two edges from L and add the latter two. Observe that the resulting graph is also in $\mathcal{L}_n(G)$.

Given an edge $e = (u, x_1)(v, y_1)$ and a cycle C of L with $e \in E(C)$, we say that $f = (u, x_2)(v, y_2)$ is (e, C)-*good* if and only if:

(i) f is not in a short cycle of L, and

(ii) $\text{dist}(f, c) \geq g$ for every $c \in V(C)$.

Starting with L_0, we construct a sequence of lifts $L_0, L_1, L_2, \dots$ such that every lift has less short cycles than the previous one. To do so, at step i we choose any cycle C_i of L_i and any edge $e_i \in E(C_i)$. Then, we choose f_i to be a (e_i, C_i)-good edge of L_i and we switch e_i and f_i. The rest of the proof consists on showing that: (1) after the switch the number of short cycles in the resulting graph has decreased, and (2) every pair (e_i, C_i) has at least one good edge f_i.

2 Existence of large girth graphs with frozen and partially-frozen colourings

Beyond the existence of lifts with large girth, our result have implications in Graph Colouring. Let G be a graph and $m \in \mathbb{N}$. The *m-recolouring graph* of G, denoted by $R_m(G)$, is the graph whose vertices are the proper m-colourings of G and two colourings are adjacent if they differ at exactly one vertex. An isolated vertex in $R_m(G)$ is called a *frozen colouring* of G, and can be understood as a colouring that admits no single-vertex recolouring that keeps its properness. The frozen terminology comes from Glauber dynamics on colourings, a Markov chain with state space $R_m(G)$ used to sample almost uniform m-colourings of G. In the dynamics, frozen states correspond to absorbing states of the chain, and impede the chain to converge to the uniform distribution. It is thus interesting to study under which conditions, such colourings may appear.

Let us first review some structural properties of $R_m(G)$. If G has maximum degree Δ, a necessary condition for the existence of frozen colourings is $m \leq \Delta + 1$. In particular, if $m = \Delta + 1$, then the graph G must be d-regular, where $d = \Delta$. In this abstract, we will restrict ourselves to the case where G is a d-regular graph and $m = d + 1$.

Feghali, Johnson and Paulusma [7] proved that $R_{d+1}(G)$ is composed of a unique connected component of size at least 2 and a number of isolated vertices (frozen colourings). Bonamy, Bousquet, and the first author [5] studied the fraction of vertices that are isolated in $R_{d+1}(G)$: when G is a large connected graph, the number of frozen colourings is exponentially smaller than the total number of colourings. This justifies that, even though the Glauber dynamics might not be irreducible, it can still be used to sample almost uniform $(d + 1)$-colourings of G.

Observe that a d-regular graph G on N vertices has a frozen colouring if and only if G is isomorphic to an n-lift of K_{d+1}, the complete graph on $d+1$ vertices, for $n(d+1) = N$. For the "if" part, one can obtain a frozen colouring of G by colouring each vertex with the corresponding vertex from K_{d+1}. For the "only if" part, any frozen colouring splits the vertex set of G into $d + 1$ independent sets of equal size, in this case n. By a simple counting argument, there are n edges within any two sets, and by the frozen condition, they form a matching. Together with (1), this shows the existence of graphs of large girth that admit a frozen colouring. As a consequence of Corollary 3, we obtain the following.

Corollary 7. *Let $n \in \mathbb{N}$, $d = d(n) \geq 3$ and $g = g(n)$ such that $(d-1)^{2g-3} = o(n)$. Then there exists a d-regular graph on n vertices and girth at least g that admits a frozen $(d + 1)$-colouring.*

Recently, Hurley and Pirot [11] studied uniformly random proper m-colourings of sparse graphs with maximum degree d in the regime $d < m \log m$. Sparsity in this setting is controlled by the girth: the larger the girth, the less density of edges in local neighbourhoods. The main concern of their paper is to understand the *shattering threshold*, the minimum number of colours that are needed for $R_m(G)$ to resemble $R_m(\mathbb{G}_{n,d})$, where $\mathbb{G}_{n,d}$ is a random d-regular graph. In this direction, they proved that a typical m-colouring of a large girth graph is not "rigid" in the following sense.

Theorem 8 ([11])**.** *Let $\epsilon > 0$ and $m \in \mathbb{N}$ large enough such that $d < (1-\epsilon)m \ln m$. If G is a graph on n vertices, maximum degree d and girth at least $\ln \ln n$, then a uniformly random proper m-colouring σ of G satisfies w.h.p.*[1]*, for all $v \in V(G)$*

(i) for all $j \in [m]$, there exists a colouring τ with $\tau(v) = j$, that differs at $O(\log^2 n)$ vertices with σ.

(ii) for all $j \in [m]$, the component of σ in $R_m(G)$ contains a colouring τ with $\tau(v) = j$.

Properties (i) and (ii) deal with the geometry of the solution space (of colourings) and are also shared with colourings of random graphs. In this direction, a natural problem is to determine which are the minimum sparsity requirements on G that ensure such properties hold. Hurley and Pirot showed that the condition $g(G) \geq \ln \ln n$ cannot be replaced by $g(G) \geq C$, for any constant $C > 0$. Here, indeed, we show that lower bound on the girth required in Theorem 8 is essentially optimal, even for $m = d + 1$.

[1]We say that a property holds *with high probability (w.h.p.)* if the probability it holds tends to 1 as $n \to \infty$.

Given an m-colouring σ of G and $v \in V(G)$, following [11], we say that v is *frozen* in σ if $\tau(v) = \sigma(v)$ for all τ in the same component of $R_m(G)$ as σ. Note that if v is frozen, then condition (ii) is not satisfied.

Proposition 9. *For every $\gamma > 0$, $d \geq 3$ and sufficiently large n_0, there exists $n \geq n_0$ and a d-regular graph G on n vertices of girth at least $\left(\frac{1}{2\ln(d-1)} - \gamma\right) \ln \ln n$ with the following property: if σ is a uniformly random proper $(d+1)$-colouring of G, w.h.p. σ has at least $n^{1-o(1)}$ frozen vertices.*

We include the proof of this proposition which is a simple application of our previous results.

Proof of Proposition 9. Let $\delta > 0$ be sufficiently small with respect to γ and d. Let $g = (1/2 - \delta) \log_{d-1} n_0$. By Corollary 7, there exists a d-regular graph G_0 on n_0 vertices of girth at least g that has a frozen colouring.

Let $\epsilon > 0$ be sufficiently small and fix n the smallest multiple of n_0 such that $n^\epsilon \geq (d+1)^{n_0}$. As δ has been chosen small enough, we have that

$$g(G_0) \geq \left(\frac{1}{2\ln(d-1)} - \gamma\right) \ln \ln n.$$

Let $k = n/n_0$ and let G be the graph composed of k vertex-disjoint copies of G_0. The uniform probability space over $(d+1)$-colourings of G is a product space of k uniform and independent probability spaces over $(d+1)$-colourings of G_0. Since G_0 admits at least one frozen $(d+1)$-colouring, the probability a uniform random $(d+1)$-colouring of G_0 is frozen is at least $p := (d+1)^{-n_0}$. It follows that the number of frozen $(d+1)$-colourings in the copies of G_0 in G stochastically dominates a Binomial random variable with k trials and probability p. By Chernoff inequality, w.h.p. the number of copies of G_0 where σ induces a frozen colourings is at least

$$\frac{k}{2(d+1)^{n_0}},$$

and the number of frozen vertices is at least

$$n_0 \cdot \frac{k}{2(d+1)^{n_0}} \geq \frac{n^{1-\epsilon}}{2}.$$

Since the choice of $\epsilon > 0$ is arbitrary, we conclude the proof of the proposition. □

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