Rainbow loose Hamilton cycles in Dirac hypergraphs

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Abstract

A meta-conjecture of Coulson, Keevash, Perarnau and Yepremyan [6] states that above the extremal threshold for a given spanning structure in a (hyper-)graph, one can find a rainbow version of that spanning structure in any suitably bounded colouring of the host (hyper-)graph. We solve one of the most pertinent outstanding cases of this conjecture, by showing that if G is an n-vertex k-uniform hypergraph with $\delta_{k-1}(G) \geq \left(\frac{1}{2(k-1)} + o(1)\right) n$, then any bounded colouring of G contains a rainbow loose Hamilton cycle.

1 Introduction

A famous theorem of Dirac [11] states that any n-vertex graph G with $\delta(G) \geq n/2$ contains a Hamilton cycle. This inspired many further results exploring the optimal minimum degree conditions for certain spanning structures in a host (hyper-)graph. This area, sometimes referred to as 'Dirac theory', is a cornerstone of modern extremal combinatorics and has flourished in recent decades due to powerful tools being developed to tackle these questions, such as the regularity method [31] and absorption [27]. In graphs, this has led to a deep understanding of the full picture with celebrated results including the minimum degree threshold for F-factors [24] (vertex disjoint copies of F covering the vertex set of the host graph) for arbitrary graphs F and the so-called Bandwidth Theorem [3] of Böttcher, Taraz and Schacht.

In hypergraphs, the situation is considerably more complex. This is, in part, due to the various ways in which one can generalise the graph case. For example, when generalising Dirac's theorem to hypergraphs, one has a range of choices as to which minimum degree condition is considered and what type of Hamilton cycle is desired. Indeed, for a k-uniform hypergraph G (k-graph for short), one can consider

$$\delta_j(G) := \min \left\{ |\{e \in E(G) : T \subset e\}| : T \in \binom{V(G)}{j} \right\},$$

for $1 \leq j \leq k-1$. The case $j = k-1$ is often called the *codegree* of the k-graph G. Likewise with Hamilton cycles, one can consider a cyclic ordering of the vertices of G and require that each edge of the Hamilton cycle occupies k consecutive vertices in the ordering and every pair of consecutive edges intersect in precisely ℓ vertices for some $1 \leq \ell \leq k-1$. Such a Hamilton cycle is called a Hamilton ℓ-cycle and when $\ell = 1$, we refer to it as a *loose* cycle, whilst the case $\ell = k-1$ is referred to as a *tight*

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cycle. Note that if an n-vertex k-graph G has a loose Hamilton cycle, then one necessarily has that $(k-1)|n$ and similar divisibility conditions hold for the other Hamilton ℓ-cycles.

In hypergraphs our understanding of minimum degree thresholds is far from complete despite a wealth of results. Indeed, even in the case that the spanning structure is a perfect matching, there are unanswered questions. We refer the reader to the survey [32] on the matter.

Whilst establishing minimum degree thresholds can be a considerable challenge, the lower bounds often follow from simple constructions that are derived to force the non-existence of the spanning structure in question. For example, the minimum codegree threshold for a loose Hamilton cycle in a k-graph is (asymptotically) $\frac{n}{2(k-1)}$ and the following example establishes the lower bound. Take n divisible by $2(k-1)$, partition $V(G) = A \cup B$ such that $|A| = \frac{n}{2(k-1)} - 1$ and take any set of k vertices that intersects A as an edge of G. Then $\delta_{k-1}(G) = |A|$ and if there was a loose Hamilton cycle in G, then in the cyclic order defining the cycle, there cannot be $2(k-1)$ consecutive vertices from B as this would contain an edge of the Hamilton cycle but no such edge exists in B. Thus there are at least $\frac{n}{2(k-1)}$ vertices in A, contradicting the size of A.

The fact that these constructions are contrived and atypical, for example having large independent sets, suggests that although one cannot weaken the respective minimum degree condition, perhaps one can strengthen the conclusion of the degree threshold. That is to say, when we are above the minimum degree threshold (we will informally refer to such (hyper-)graphs as being 'Dirac') with respect to a given spanning structure, the Dirac (hyper-)graph is in fact *robust* with respect to containing that spanning structure. Various results of this flavour have been established, in particular in the context of Dirac's condition for Hamilton cycles, see the nice survey of Sudakov [30]. For example, it has been shown that there are in fact *many* Hamilton cycles above the extremal threshold [10], as well as many edge-disjoint Hamilton cycles [9]. In this paper, we will consider a notion of robustness related to finding *rainbow* spanning structures in any bounded edge colouring of the Dirac (hyper-)graph. This is motivated by the classical study of rainbow spanning structures in certain colourings of graphs.

1.1 Rainbow spanning structures

A subgraph H of an edge coloured graph G is said to be *rainbow* if each of the edges of H is a different colour. Rainbow subgraphs appeared early on in combinatorics via connections with design theory. Indeed, already Euler [15] was interested in transversals in Latin squares, which is a collection of entries in the Latin square with distinct rows, columns and symbols. Viewing an $n \times n$ Latin square as an edge colouring of a complete bipartite graph, with parts corresponding to columns and rows and colour classes corresponding to symbols, a transversal becomes a rainbow matching. Several beautiful conjectures were posed in design theory, that are only now being solved by heavily utilising connections to rainbow spanning subgraphs. Indeed, perhaps the most famous such conjecture, known as the Ryser-Brualdi-Stein conjecture [4, 28, 29] states that every $n \times n$ Latin square has a transversal of size at least $n-1$ and one of size n when n is odd. The first part of this (establishing the existence of transversals of size $n-1$) has only recently been solved by Montgomery [25]. Translating to colourings of graphs, the Ryser-Brualdi-Stein conjecture asserts that one can always find an (almost) perfect rainbow matching. Here, the conditions of the Latin square are equivalent to the colouring of $K_{n,n}$ having n colours and being *proper*, that is, there are no two edges of the same colour at a vertex.

From a graph theoretic perspective one can ask more generally what conditions on a colouring of a host graph guarantee the existence of a rainbow (almost) spanning structure of interest. The Ryser-Brualdi-Stein conjecture, as well as a host of other conjectures inspired by design theory, suggest that the colouring being proper is enough. In search for other conditions, researchers noted that a colouring being proper is equivalent to saying that the colouring is *locally bounded*, that is, at each vertex we see every colour at most once, or more generally, a bounded number of times. One can also then consider *globally bounded* conditions where we bound the size of each colour class.

An early example of interest in rainbow structures under global bounded conditions on colouring was due to Erdős and Stein (see [13]) who asked whether there is some constant $c > 0$ such that any colouring of K_n with at most cn edges of each colour contains a rainbow Hamilton cycle. This was then explicitly conjectured by Hahn and Thomassen [18] and, after several results towards the conjecture, was solved by Albert, Frieze and Reed [1]. A generalisation to hypergraph Hamilton cycles was then given by Dudek, Frieze and Ruciński [12]. There has been a wealth of similar results studying different spanning structures.

One may wonder how optimal these results are. For example, note that the result of Albert, Frieze and Reed is tight up to the choice of constant $c > 0$; a value of $c < 1/2$ is certainly necessary as otherwise there may not be enough colours to have a rainbow Hamilton cycle. In the setting of perfect matchings in complete bipartite graphs, Stein [29] boldly conjectured that the condition of being proper could be dropped and replaced by each colour class simply having size n. This turned out to be false with Pokrovskiy and Sudakov [26] recently giving a construction with n edges of each colour and no rainbow transversal bigger than $n - \Omega(\log n)$. This shows that in this setting, a colouring having a global bound of n edges of each colour is not enough to guarantee the desired rainbow matching of size $n - 1$. However, in what was a hugely influential paper and the first in this area of finding rainbow structures in globally bounded colourings, Erdős and Spencer [14] showed that any colouring of $K_{n,n}$ with at most $\frac{n}{16}$ edges of each colour contains a rainbow perfect matching.

1.2 Rainbow structures in Dirac (hyper-)graphs

The vast majority of results concerning rainbow spanning substructures in bounded (and proper) colourings have focused on the case where the host graph is a complete (hyper-)graph or complete bipartite graph. When considering other possible host graphs, Dirac graphs arise naturally. Indeed, in order to contain a rainbow copy of a desired spanning subgraph in any bounded colouring, the host graph certainly needs to contain copies of that subgraph and so imposing the existence of such subgraphs through minimum degree conditions gives a natural class of candidate host graphs. This perspective was first considered by Cano, Perarnau and Serra [5] who showed that one can find a rainbow Hamilton cycle in any globally $o(n)$-bounded colouring of G when G is either an n-vertex graph or a balanced bipartite graph with n vertices in each part, and such that G has minimum degree $\delta(G) \geq (1 + o(1))\frac{n}{2}$. The asymptotic minimum degree condition was then replaced to give an exact minimum degree condition $\delta(G) \geq \frac{n}{2}$ by Coulson and Perarnau, first in the bipartite case [7] and then in the non-bipartite case [8] as in Dirac's original theorem. These results thus give evidence of robustness for the extremal thresholds for Hamilton cycles. Note also that in the bipartite case, these results can be seen as a direct strengthening of the result of Erdős and Spencer [14], allowing for host graphs that are not complete (at the expense of a potentially worse constant for the boundedness).

Further examples of these types of results came from Coulson, Keevash, Perarnau and Yepremyan [6] who proved that (asymptotically) above the minimum degree for a given (hyper-)graph F-factor, one finds a rainbow F-factor in any suitably bounded colouring, and from Glock and Joos [16] who gave a rainbow version of the famous blow-up lemma [22], which allowed them to give results of this flavour in considerable generality for graphs, in particular providing a rainbow version of the bandwidth theorem [3]. We remark that a nice feature of the work of [6] is that they could establish such a result, even in cases where the minimum degree threshold has not yet been determined.

All of these results provide evidence of a general phenomenon and caused Coulson, Keevash, Perarnau and Yepremyan [6] to explicitly give the "meta-conjecture" that once one is above the extremal threshold for a given spanning structure, rainbow copies of that structure can be found in any suitably bounded colouring of the Dirac graph. Our main result provides further evidence for this conjecture, by establishing that this is the case for loose Hamilton cycles in hypergraphs.

Theorem 1. *For any $2 \leq k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\mu > 0$ such that for any sufficiently large $n \in (k-1)\mathbb{N}$, the following holds. If G is a k-graph with $\delta_{k-1}(G) \geq (1+\varepsilon)\frac{n}{2(k-1)}$ and $\chi : E(G) \to \mathbb{N}$ is colouring of G with at most μn^{k-1} edges of each colour and at most μn edges of each colour containing* *any given $(k-1)$-set of vertices, then there exists a rainbow loose Hamilton cycle.*

Theorem 1 provides a first generalisation of the result of Coulson and Perarnau [8] to the hypergraph setting. Note that the minimum degree condition is asymptotically tight due to the construction discussed above. The fact that hypergraphs with minimum codegree at least $(1+o(1))\frac{n}{2(k-1)}$ contain loose Hamilton cycles was proven originally by Kühn and Osthus [23] for $k=3$ and for general k by Hàn and Schacht [19] and independently by Keevash, Kühn, Mycroft and Osthus [20]. Our result can thus be seen as a direct strengthening of these results, providing robustness. We remark that the tight minimum codegree threshold (without the $o(1)$ factor) for the existence of a loose Hamilton cycle is unknown and seems to be a considerable challenge.

Note also that the global bound in Theorem 1 is also tight, up to the choice of the constant μ. Indeed, some global bound of the order of n^{k-1} is needed to guarantee enough colours. The local bound in Theorem 1, requiring each $(k-1)$-set to be in at most μn edges of any given colour, is rather weak in comparison to requiring a colouring to be proper, for example. It is unclear whether this local bound is in fact necessary. Indeed this condition arises as somewhat of a technicality within the proof which nonetheless seems hard to bypass. This local boundedness condition was also present in the previous result of Coulson, Keevash, Perarnau and Yepremyan [6] on rainbow factors and it can be shown to be necessary when dealing with clique factors or tight Hamilton cycles (for which it remains an open question to prove an analogue of Theorem 1). At the cost of this extra local bound, Theorem 1 strengthens the previously mentioned work of Dudek, Frieze and Ruciński [12] who proved Theorem 1 in the case that the host hypergraph G is complete. Finally, we mention a result of Antoniuk, Kamčev and Ruciński [2] who showed that under the same assumption that $\delta_{k-1}(G) \geq (1+o(1))\frac{n}{2(k-1)}$, any colouring in which each vertex is contained in at most $o(n^{k-1})$ edges of the same colour results in a Hamilton loose cycle that is properly coloured, that is, the Hamilton cycle does not contain incident edges of the same colour. Our result strengthens the conclusion by guaranteeing a rainbow loose Hamilton (which is in particular proper) at the cost of adopting both a local bound and a global bound for the colouring, the latter being necessary for the rainbow setting, as previously discussed.

2 A proof overview

The *lopsided local lemma*, originally introduced by Erdős and Spencer [14] in the context of rainbow perfect matchings in $K_{n,n}$, provides a general tool for finding rainbow spanning structures in bounded colourings of host graphs. The setup works by taking a uniformly random copy of the desired spanning structure and defining bad events based on two edges of the same colour appearing in this random sample. This setting does *not* have limited dependence between our bad events and so the original local lemma cannot be used to show that the uniform copy is rainbow with some positive probability. Nonetheless, Erdős and Spencer showed that the desired conclusion of the local lemma indeed holds if we can bound the amount of *negative dependence* between bad events. In the setting of complete (bipartite) graphs, one can carefully count copies of the desired spanning structure subject to certain bad events not taking place, allowing calculations of conditional probabilities necessary to show such negative dependence.

When the host graph is no longer complete, precise counts of spanning structures are no longer accessible. The key idea in the initial works [5, 7, 8] in Dirac host graphs, is that one can still estimate the required conditional probabilities necessary, by applying a "switching method". Here one locally alters some fixed copy of the spanning structure in a way that maintains some fixed events that we want to condition on. If we can find many ways of performing valid switchings, we can provide upper bounds on conditional probabilities to show that there is enough negative dependence in the collection of bad events for the lopsided local lemma. This switching approach was then used again in the work of Coulson, Keevash, Perarnau and Yepremyan [6] finding rainbow F-factors. Their key innovation was that one can find many switchings via probabilistic methods. They take a random sample of the vertex set (in fact, a random sample of copies of F in the factor we are switching from) and show that with probability bounded away from 0 one can perform the switch within this random set, obtaining a new factor where some copies have been reshuffled. This translates to having many subsets providing valid switches and opens up the power of the probabilistic method to prove the existence of valid switchings. Indeed, with high probability, the sampled vertex set will inherit many nice properties of the host graph, in particular the minimum degree condition. After some work (to ensure the switching is valid), this allows the authors of [6] to apply the existence of a sub-F-factor in the random vertex set as a black box, using that the minimum degree condition is satisfied.

Our proof again follows this template and we will again use random samples to provide many switchings, setting up an application of the lopsided local lemma. There is one major hurdle in our setting as opposed to F-factors though, which comes from the fact that we are now dealing with *connected* spanning structures. This means that we cannot locally adjust our copy within the random set independently of the rest of the spanning structure. This hurdle was noted also in [2] and means that one can no longer use black box results in the random set of vertices. In order to overcome this, we use absorption techniques to rebuild the loose Hamilton cycle in the random set in such a way that it provides a valid switching. In more detail, we use an absorbing strategy due to Hàn and Schacht [19] which gives an absorbing structure as well as a connecting lemma that we can use to piece back together our loose Hamilton cyle.

To our knowledge, this is a first example of absorption being used in the context of the local lemma and we find it a nice feature of our proof that it simultaneously incorporates two of the most powerful methods in modern extremal and probabilistic combinatorics.

3 Further directions

We believe our method of using the lopsided local lemma in conjunction with absorption techniques has the potential to prove more results in the setting of robustness via rainbow structures in bounded colourings. In particular, for different Hamilton ℓ-cycles in hypergraphs under different minimum j-degree conditions, whenever there is an existing proof for the existence of the cycle that appeals to absorption techniques, there is a hope to apply our framework. This is reminiscent of recent work in the setting of transversal spanning structures [17] and robustness via percolation [21], where they provide certain 'absorption-necessary' conditions in order to give general results that follow from the previous work in establishing extremal thresholds, in particular covering many different types of Hamilton cycle and minimum degree conditions. The full power of our approach will be explored in a forthcoming journal version of this extended abstract.

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