The flexibility among 3-decompositions

Irene Heinrich *1 and Lena Volk †1

[1]Dept. of Mathematics, Technische Universität Darmstadt, 64289 Darmstadt, Germany

Abstract

The 3-decomposition conjecture, postulated by Hoffmann-Ostenhof in 2011, is a major open question about the structure of cubic graphs: *Can the edge set of every cubic graph be decomposed into a spanning tree, a disjoint union of cycles, and a matching?* To date, the conjecture remains wide open. Towards a deeper structural understanding of 3-decompositions, we investigate the set of all 3-decompositions of a graph as a whole. On the one side, we provide a graph class that displays extremal behaviour: up to isomorphism, only one 3-decomposition exists. On the other side, we show that in general, 3-decompositions are more flexible. This contrasts the existing approaches which focus on the construction of precisely one decomposition of the considered graph. We exploit these insights towards a verification of the 3-decomposition conjecture on Bilu-Linial expanders.

1 Introduction

All graphs in this paper are simple and finite. A *3-decomposition* of a cubic graph G is a triple (T, C, M) of subgraphs of G where T is a spanning tree of G, C is 2-regular, and M is a matching such that $\{E(T), E(C), E(M)\}$ is a partition of $E(G)$. (See Figure 1 for examples of 3-decompositions.) The 3-decomposition conjecture, postulated by Hoffmann-Ostenhof [6], is a central open question about the structure of cubic graphs.

3-Decomposition Conjecture. *Every connected cubic graph has a 3-decomposition.*

The 3-decomposition conjecture has received great interest, and numerous results verify the conjecture on subclasses (e.g., planar [7], treewidth-3 [5], pathwidth-4 [2], and claw-free graphs [1]). Li and Cui [10] proved that the following weaker variant of the 3-decomposition conjecture is true: Every connected cubic graph can be decomposed into a spanning tree, a disjoint union of cycles, and a disjoint union of paths of length at most 2. There is ample literature on 3-decompositions when the considered

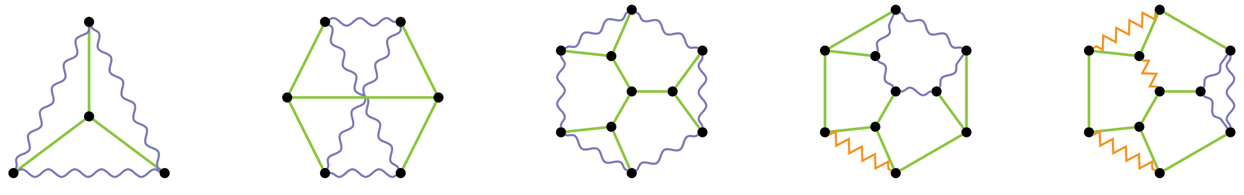


Figure 1: 3-decompositions of K_4, $K_{3,3}$, and three distinct 3-decompositions of the tricorn graph G_T. Spanning tree edges are straight and green, cycle edges are wavy blue, and matching-edges are zigzag-shaped and orange. It holds $\min_{\mathrm{MATCH}}(G_T) = 0$ and $\max_{\mathrm{MATCH}}(G_T) = 3$.

*Email: heinrich@mathematik.tu-darmstadt.de. The research leading to these results has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (EngageS: grant agreement No. 820148).

†Email: volk@mathematik.tu-darmstadt.de.

graph G admits one of the following two extremes: a Hamiltonian path (a tree maximizing the number of degree-2 vertices) or a HIST [6] (a spanning tree which is *homeomorphically irreducible*, i.e., free of degree-2 vertices). However, little is known about the structure of the set of *all* 3-decompositions of a cubic graph. Towards a deeper structural understanding of cubic graphs we analyze the set of all 3-decompositions of a graph (class). We focus on the following three questions.

Q 1. *Are there graphs with a unique 3-decomposition?*

Consider the two graph invariants

$$\begin{aligned}\min\nolimits_{\mathrm{MATCH}}(G) &\coloneqq \min\{\|M\|\colon (T,C,M) \text{ is a 3-decomposition of } G\}, \text{ and}\\ \max\nolimits_{\mathrm{MATCH}}(G) &\coloneqq \max\{\|M\|\colon (T,C,M) \text{ is a 3-decomposition of } G\},\end{aligned}$$

where $\|\cdot\|$ denotes the size (i.e., the number of edges) of a graph.

Q 2. *Which graphs (or graph classes) are extremal with respect to* $\min_{\mathrm{MATCH}}$ *and* $\max_{\mathrm{MATCH}}$*, respectively? How flexible is the set of all 3-decompositions of a graph with respect to the number of matching edges it contains?*

Q 3. *How can we exploit the observed flexibility among 3-decompositions towards proving the 3-decomposition conjecture?*

Our contribution. We positively answer Question 1 by providing an infinite class of graphs with the property that each graph in the class has a unique 3-decomposition up to isomorphism (Theorem 4). It is noteworthy that all graphs in this class have a HIST and it is known that a HIST of a cubic graph naturally corresponds to a 3-decomposition [6]. Hence, we further investigate for which graphs there exist HIST-free 3-decompositions. Assuming the 3-decomposition conjecture to hold, we prove that every graph of connectivity 2 has a HIST-free 3-decomposition (Theorem 5). We used the computer to verify that apart from K_4 and $K_{3,3}$ every 3-connected cubic graph of order at most 20 has a 3-decomposition without a HIST (Theorem 6).

Concerning Question 2, we prove that there exists a family of graphs $(H_n)_{n\in\mathbb{N}}$ with $\min_{\mathrm{MATCH}}(H_n) = 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \max_{\mathrm{MATCH}}(H_n) = \infty$ (Proposition 8). We complement this result by proving the existence of two other graph families $(G_n)_{n\in\mathbb{N}}$ and $(G'_n)_{n\in\mathbb{N}}$ that satisfy $\lim_{n\to\infty} \max_{\mathrm{MATCH}}(G_n) = 0$ (Theorem 4) and $\lim_{n\to\infty} \min_{\mathrm{MATCH}}(G'_n) = \infty$ (Proposition 7).

We give a partial answer to Question 3 in Section 5 where we highlight that the flexibility of 3-decompositions can be exploited in order to provide a tighter analysis of 3-decompositions of Bilu-Linial expanders. This broadens our understanding of 3-decompositions since expander graphs show completely different behavior compared to the previously studied classes.

Due to space restrictions, some of the proofs are omitted or shortened to proof sketches.

Further related work. The recent results on the 3-decomposition conjecture are surveyed in the introduction of [2]. Hoffmann-Ostenhof, Noguchi, and Ozeki studied the existence of HISTs in cubic graphs [8]. Deciding whether a graph allows for a HIST is in general an intractable problem, which remains intractable even if the input is restricted to the class of cubic graphs [4].

2 Preliminaries

For two integers a and b we set $[a,b] \coloneqq \{a, a+1, \ldots, b\}$. We denote the path of order n by P_n, the complete graph of order n by K_n, and the complete bipartite graph with one part on n vertices and the other part on m vertices by $K_{n,m}$. The $\dot{K}_4$ is a graph obtained from K_4 by subdividing precisely one edge. Analogously, the $\dot{K}_{3,3}$ is obtained by subdividing an edge of the $K_{3,3}$. If u and v are vertices of a tree T, then uTv denotes the unique u-v-path in T. For a graph G and an edge subset $E' \subseteq E(G)$

we set $G[E']$ to be the graph with edge set E' and vertex set $\{v \in V(G)\colon \exists u \in V(G)\colon uv \in E'\}$. A non-empty graph G is *k-connected* (*k-edge connected*) if for each two distinct vertices u and v of G there are at least k internally vertex-disjoint (edge-disjoint) u-v-paths in G. The maximum number $k \in \mathbb{N}$ such that G is k-connected (k-edge connected) is the *connectivity* (*edge connectivity*) of G. In contrast to general graphs, the connectivity and the edge-connectivity of a cubic graph are equal. If $E' \subseteq E(G)$ such that $G[E(G) \setminus E']$ has more components than G, then E' is an *$|E'|$-edge separator*, otherwise we call $G[E']$ *non-separating*. Let $E' \subseteq E(G)$. If there exists a bipartition $\{U, W\}$ of $V(G)$ such that $E' = \{uw \in E(G)\colon u \in U, w \in W\}$, then E' is a *cut set* of G. A *bridge* is a 1-edge separator. If G is cubic and has a 3-decomposition (T, C, M), then G is *3-decomposable*.

Lemma 1. *Let G be a cubic graph with a 3-decomposition (T, C, M).*

1. $\|G\| = {}^3\!/_2|V(G)|$ *and* $\|C\| + \|M\| = \|G\| - \|T\| = {}^{|V(G)|}\!/_2 + 1$.
2. *C and M are non-separating subgraphs of G.*
3. *Each vertex $v \in V(G)$ is either a degree-3 vertex of T, or a degree-2 vertex of T and contained in M, or a degree-1 vertex of T and contained in C. In particular,* $\|C\| \geq 3$

Observation 2 (Reformulation of [8], Theorem 2)**.** *If G is a cubic graph, then* $\min_{\mathrm{MATCH}}(G) = 0$ *if and only if there exists a HIST T of G, which is the case precisely if $(T, G[E(G) \setminus E(T), \emptyset])$ is a 3-decomposition of G.*

Lemma 3. *If G is a 3-decomposable graph and* $\ell := \min\{\|C\|\colon C$ *is a non-separating cycle in* $G\}$*, then*

$$0 \leq \min_{\mathrm{MATCH}}(G) \leq \max_{\mathrm{MATCH}}(G) \leq {}^1\!/_2|V(G)| + 1 - \ell \leq {}^1\!/_2|V(G)| - 2.$$

3 Graphs with unique 3-decompositions

In this section, we tackle Question 1. In fact, already among the smallest cubic graphs there are two examples of graphs with a unique 3-decomposition up to isomorphism: K_4 and $K_{3,3}$. The K_4 decomposes into a $K_{1,3}$, a 3-cycle, and an empty matching; the $K_{3,3}$ decomposes into a tree known as the *H-graph*, a 4-cycle, and an empty matching (see Figure 1). Observe that, in accordance with Observation 2, each of the two trees is a HIST. We argue that the decompositions are unique: By Lemma 1.3 each of the decompositions contains a cycle. Observe that a shortest cycle in K_4 is a 3-cycle and each two 3-cycles of K_4 can be mapped to each other by an automorphism of K_4. The remaining edges form a $K_{1,3}$. In particular, no larger cycle can be part of a 3-decomposition of K_4. The uniqueness of the decomposition of $K_{3,3}$ can be proven in analogy to this. In fact, there are infinitely many graphs with this property:

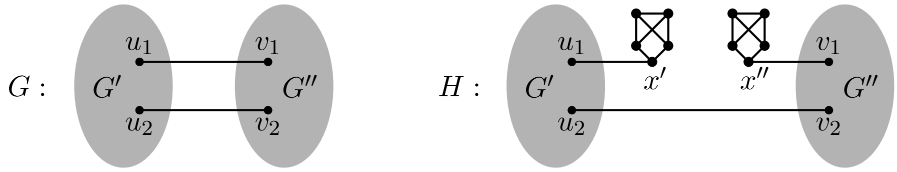
Theorem 4. *There exists an infinite family $\mathcal{G}$ of graphs which have precisely one 3-decomposition up to isomorphism. Further,* $\min_{\mathrm{MATCH}}(G) = \max_{\mathrm{MATCH}}(G) = 0$ *for all $G \in \mathcal{G}$.*

Proof sketch. The class $\mathcal{T}$ of homeomorphically irreducible subcubic trees contains infinitely many non-isomorphic trees. Let $\mathcal{G}$ be the family of cubic graphs obtained by the following construction: Let $T \in \mathcal{T}$. For each leaf ℓ of T let K^ℓ be either a copy of $\dot{K}_4$ or $\dot{K}_{3,3}$. Take the disjoint union of T and the graphs in $\{K^\ell\colon \deg_T(\ell) = 1\}$ and identify the degree-2 vertex of K^ℓ with ℓ for each leaf ℓ of T. The ingredients for the uniqueness proof are as follows: For a graph $G \in \mathcal{G}$ all edges of the corresponding tree $T \in \mathcal{T}$ are bridges of the construction, further each appended $\dot{K}_4$ or $\dot{K}_{3,3}$ has (up to isomorphism) precisely one non-separating cycle. The resulting decomposition is free of matching-edges. $\square$

The situation observed at the beginning of this section (the only option of obtaining a 3-decomposition corresponds to a HIST) never occurs in the setting of connectivity-2 graphs if the 3-decomposition conjecture holds:

Theorem 5. *If the 3-decomposition conjecture holds, then every cubic graph with connectivity 2 has a 3-decomposition with a non-empty matching.*

Proof. We show the following stronger claim: If the 3-decomposition conjecture holds, then every connected cubic graph which has a 2-edge separator of non-incident edges has a 3-decomposition with a non-empty matching. The theorem follows immediately from this claim since if two incident edges e_1, e_2 form a 2-edge separator of a cubic graph, then the unique edge incident to e_1 and e_2 is a bridge (and, hence, the connectivity is at most 1). Assume that the 3-decomposition conjecture holds and let G be a cubic graph with a HIST T and a 2-edge separator of non-incident edges $\{u_1v_1, u_2v_2\}$. The graph $G \setminus \{u_1v_1, u_2v_2\}$ has precisely two components G' and G''.



Set $C \coloneqq G[E(G) \setminus E(T)]$ and consider the 3-decomposition $(T, C, \emptyset)$ of G. Since $\{u_1v_1, u_2v_2\}$ is a separator we obtain $\{u_1v_1, u_2v_2\} \cap E(C) = \emptyset$ and, hence $\{u_1v_1, u_2v_2\} \subseteq E(T)$. Precisely one of the following situations occurs: $u_1Tu_2 \subseteq G'$ or $v_1Tv_2 \subseteq G''$. We may assume that $u_1Tu_2 \subseteq G'$.

We construct a graph H as follows: add two new vertices x' and x'' to G, remove u_1v_1, and add the edges u_1x' and $x''v_1$. Take the disjoint union of this graph with two copies K' and K'' of $\dot{K}_4$ and identify x' (resp. x'') with the degree-2 vertex of K' (resp. K''). The resulting graph H has a 3-decomposition (T_H, C_H, M_H) by assumption. Since u_1v_1 and u_2v_2 are not incident $\|u_1Tu_2\| \geq 1$ and we may choose an edge $e \in E(u_1Tu_2)$. Now, merge the 3-decomposition induced by the HIST and the one of H together to one for G. Take the decomposition from H in G'' and in G' a slight modification of the decomposition induced by the original HIST: Remove the edge e from the spanning tree part of T in G' to disconnect u_1 and u_2 in the spanning forest in G' and instead connect them via the spanning tree T_H in G'', which connects v_1 and v_2. We may add e to the matching since the decomposition used on G' so far had an empty matching. More formally $((T \cap G') \cup (T_H \cap G'') \cup (u_1, v_1) \cup (u_2, v_2) \setminus \{e\}, (C \cap G') \cup (C_H \cap G''), \{e\} \cup (M_H \cap G''))$ is a 3-decomposition with a non-empty matching for G. ☐

Theorem 6. *Apart from K_4 and $K_{3,3}$, every 3-connected cubic graph of order at most 20 has a 3-decomposition with a non-empty matching*[1].

4 Flexibility among 3-decompositions

Proposition 7. *For every $n \in \mathbb{N}$ there exists a 2-connected cubic graph G'_n with $\min_{\text{MATCH}}(G'_n) = n$.*

Proof sketch. We refrain from giving a technical description of G'_n and refer to Figure 2 for the construction and a 3-decomposition of G'_n with precisely n matching-edges. In particular, the graph G'_n is 3-decomposable and $\min_{\text{MATCH}}(G'_n) \leq n$. Assume that (T_n, C_n, M_n) is a 3-decomposition of G'_n. Observe that the only non-separating cycles in G'_n are the four triangles (in Figure 2: two triangles on the left and two triangles on the right of the drawing). At most one of the two left triangles and at most one of the two right triangles can be contained in C_n since C_n is a disjoint union of separating cycles by Lemma 3. Further, since C_n is non-empty we obtain $\|C_n\| \in \{3, 6\}$. From Lemma 1 follows $\|C_n\| + \|M_n\| = n + 6$ and with this, $\|M_n\| \geq n$. ☐

Theorem 8. *For every odd number $n \in \mathbb{N}$ there exists a cubic graph H_n with $\min_{\text{MATCH}}(H_n) = 0$ and $\max_{\text{MATCH}}(H_n) = n$. Further, there exists a 3-decomposition of H_n with $n - 3l$ edges in the matching for every $l \in [0, {}^{(n+1)}/_4]$.*

[1] https://gitlab.rlp.net/obachtle/reductions-for-the-3-decomposition-conjecture, March 2024.

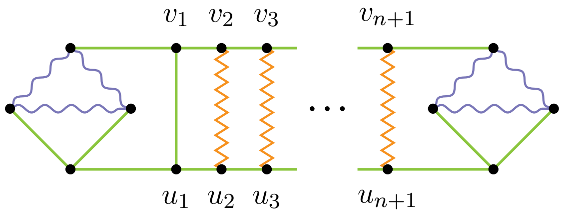


Figure 2: The graph G'_n is a 2-connected cubic graph with $\min_{\mathrm{MATCH}}(G'_n) = n$.

Proof sketch. We only discuss the two extreme cases in this sketch. Fix an odd number $n \in \mathbb{N}$ and set $k := {}^{(n+3)}/_2$. Let $P = v_1 v_2 \dots v_k$ be the k-vertex path. Let Q^1 and Q^k be two copies of P_3 and let $K^2, K^3, \dots, K^{k-1}$ be $k-2$ copies of $K_{1,3}$. Take the disjoint union of P, Q^1, Q^k, and all K^i for $i \in [2, k-1]$. Now, identify the degree-2 vertex of Q_1 (resp. Q_k) with v_1 (resp. v_k). Further, for each $i \in [2, k-1]$ identify v_i with a degree-1 vertex of K^i. The resulting tree T has $2k-2$ degree-3 and $2k$ degree-1 vertices. Choose a planar embedding of T and connect the leaves of T by the outer facial cycle. Then, the resulting graph H_n is cubic and has the HIST T. Thus, $\min_{\mathrm{MATCH}}(H_n) = 0$. Further, $\max_{\mathrm{MATCH}}(H_n) = n$ since the shortest non-separating cycle is of length 3 and a 3-decomposition with n matching-edges can be obtained as follows: Let C_n be the triangle induced by the vertices of Q^1 in H_n. The following edges form the matching M_n: for $i \in [2, k-1]$ the edge of the outer face joining two vertices of K^i and the edge joining the degree-3 vertex of the K^i to P, and the edge of the outer face joining two vertices of Q^k. Let $T_n = H_n[E(H_n) - E(C_n) - E(M_n)]$. The triple (T_n, C_n, M_n) is a 3-decomposition of H_n with $\|M_n\| = n$. For $n = 3$ the 3-decompositions are depicted in the third and the fifth graph in Figure 1. $\square$

5 3-Decompositions of Bilu-Linial Expanders

Bilu and Linial [3] give a concrete construction for a family of expander graphs by a series of lifting operations associated to random signings. See [9] for a survey on expanders. In the following, we investigate how 3-decompositions can be lifted. This illustrates how exploiting the flexibility of 3-decompositions, yields a fruitful approach to verify the 3-decomposition conjecture for more classes of graphs. The *2-lift* of a graph G equipped with a *signing* $s\colon E(G) \to \{-1, 1\}$ is the graph $\mathrm{lift}(G, s)$ with

$$V(\mathrm{lift}(G,s)) = \{v_0 \colon v \in V(G)\} \cup \{v_1 \colon v \in V(G)\},$$
$$E(\mathrm{lift}(G,s)) = \bigcup_{uv \in s^{-1}(1)} \{u_0v_0, u_1v_1\} \ \cup \bigcup_{uv \in s^{-1}(-1)} \{u_0v_1, u_1v_0\}.$$

The vertices v_0 and v_1 are *fibers* of v and $\deg_{\mathrm{lift}(G,s)}(v_0) = \deg_{\mathrm{lift}(G,s)}(v_1) = \deg_G(v)$. For a subgraph H of G, we set $\mathrm{lift}(H, s) := \mathrm{lift}(H, s|_{E(H)})$. Observe that $\mathrm{lift}(H, s)$ is a subgraph of $\mathrm{lift}(G, s)$. The *signing* of a path $P \subseteq G$ is $s(P) := \prod_{e \in E(P)} s(e)$. In general, the existence of HISTs is not preserved by 2-lifts: If G is a cubic graph with a HIST T and $s \equiv -1$, then $\mathrm{lift}(G, s)$ is bipartite and $|V(\mathrm{lift}(G, s))|$ is a multiple of 4. It follows with [8, Corollary 3] that $\mathrm{lift}(G, s)$ does not have a HIST. In contrast to this, 3-decompositions can be lifted under certain preconditions on the signing and the matching-edges.

Note that the lift of a connected graph is not necessarily connected again. E.g., if G is connected and $s \equiv 1$, then $\mathrm{lift}(G, s)$ is isomorphic to the disjoint union of two copies of G. The assumptions of Theorem 10 ensure that the considered lift is connected. In the following, we characterize signings which yield a disconnected lift in order to show that 3-decompositions can be lifted in this case.

Lemma 9. *Let G be a connected graph with a signing s. The following are equivalent:*
1. *$\mathrm{lift}(G, s)$ is disconnected.*
2. *$s^{-1}(-1)$ is empty or a cut set of G.*
3. *$\mathrm{lift}(G, s)$ is isomorphic to two disjoint copies of G.*

In particular, if G is 3-decomposable and $\operatorname{lift}(G,s)$ *is disconnected, then each of the two components of* $\operatorname{lift}(G,s)$ *is 3-decomposable.*

Theorem 10. *Let (T,C,M) be a 3-decomposition of a cubic graph G with a signing s. If there exists $xy \in E(M)$ such that $s(xy) = -1$ and $s(xTy) = 1$, then the following is a 3-decomposition of* $\operatorname{lift}(G,s)$*:*

$$(\operatorname{lift}(T,s) + x_0y_1, \operatorname{lift}(C,s), \operatorname{lift}(M,s) - x_0y_1).$$

A random variable $s\colon E(G) \to \{-1,1\}$ is a *random signing* of G if the sign of each edge is chosen uniformly at random.

Lemma 11. *Let G be a graph with a 3-decomposition (T,C,M). If s is a random signing of G, then*

$$\mathbb{P}\left[\exists xy \in E(M)\colon s(xy) = -1 \wedge s(xTy) = 1\right] = 1 - (3/4)^{\|M\|}.$$

Corollary 12. *If G is a cubic graph and s is the random signing on G, then the probability that the construction of Theorem 10 yields a 3-decomposition of* $\operatorname{lift}(G,s)$ *is maximized if the considered 3-decomposition (T,C,M) of G satisfies $\|M\| = \max_{\mathrm{MATCH}}(G)$.*

When iteratively applying the lifting operation, the number of edges in the matching m_k of the k-th lift G_k is $2^k(m_0 - 1) + 1$. Thus, the probability that iteratively applying Theorem 10 yields a 3-decomposition of G_k is at least $\prod_{l=0}^{k-1}\left(1 - (3/4)^{m_l}\right)$.
One can significantly improve this bound using that each lift yields at least two valid 3-decompositions (use x_1y_0 instead of x_0y_1 in Theorem 10) and, hence, two distinct possible edges in the matching.

6 Further research

The most pressing question is whether the flexibility of 3-decompositions can be exploited in order to prove the 3-decomposition conjecture on expander graphs or on symmetric graphs. Further, it is desirable to classify all graphs which have a unique 3-decomposition up to isomorphism.

References

[1] E. Aboomahigir, M. Ahanjideh, and S. Akbari. Decomposing claw-free subcubic graphs and 4-chordal subcubic graphs. *Discret. Appl. Math.*, 296:52–55, 2021.

[2] O. Bachtler and I. Heinrich. Reductions for the 3-decomposition conjecture. In *LAGOS*, volume 223 of *Procedia Computer Science*, pages 96–103. Elsevier, 2023.

[3] Y. Bilu and N. Linial. Constructing expander graphs by 2-lifts and discrepancy vs. spectral gap. In *FOCS*, pages 404–412. IEEE Computer Society, 2004.

[4] R. J. Douglas. NP-completeness and degree restricted spanning trees. *Discret. Math.*, 105(1-3):41–47, 1992.

[5] I. Heinrich. *On Graph Decomposition: Hajos' Conjecture, the Clustering Coefficient, and Dominating Sets.* PhD thesis, TU Kaiserslautern, 2019.

[6] A. Hoffmann-Ostenhof. *Nowhere-zero flows and structures in cubic graphs.* PhD thesis, University Vienna, 2011.

[7] A. Hoffmann-Ostenhof, T. Kaiser, and K. Ozeki. Decomposing planar cubic graphs. *J. Graph Theory*, 88(4):631–640, 2018.

[8] A. Hoffmann-Ostenhof, K. Noguchi, and K. Ozeki. On homeomorphically irreducible spanning trees in cubic graphs. *J. Graph Theory*, 89(2):93–100, 2018.

[9] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc.*, 43(4):439–561, 2006.

[10] R. Li and Q. Cui. Spanning trees in subcubic graphs. *Ars Comb.*, 117:411–415, 2014.