Extending the Continuum of Six-Colorings*

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Abstract

We present two novel six-colorings of the Euclidean plane that avoid monochromatic pairs of points at unit distance in five colors and monochromatic pairs at another specified distance d in the sixth color. Such colorings have previously been known to exist for $0.41 < \sqrt{2}-1 \leq d \leq 1/\sqrt{5} < 0.45$. Our results significantly expand that range to $0.354 \leq d \leq 0.657$, the first improvement in 30 years. The constructions underlying this notably were derived by formalizing colorings suggested by a custom machine learning approach.

1 Introduction

The Hadwiger-Nelson problem asks for the smallest number of colors needed to color the points of the Euclidean plane $\mathbb{E}^2$ without any two points a unit distance apart having the same color. Viewing the plane as an infinite graph, with an edge between any two points if and only if the distance between them is 1, motivates why this number is also referred to as the *chromatic number of the plane* and denoted by $\chi(\mathbb{E}^2)$. The problem goes back to 1950 and has since become one of the most enduring and famous open problems in combinatorial geometry and graph theory. For an extensive history of the problem and results related to it, we refer the reader to Jensen and Toft [7] as well as Soifer [9, 17].

By the de Bruijn–Erdős theorem [1], and therefore assuming the axiom of choice, the problem is equivalent to finding the largest possible chromatic number of a finite unit distance graph, that is a graph that can be embedded into the plane such that any two vertices are adjacent if and only if the corresponding points are at unit distance. The triangle is one obvious such graph, giving a lower bound of 3, and the Moser spindle [8] is the most famous example of a graph giving a lower bound of 4. There had been no improvement to that lower bound since 1950 until de Grey famously established that $\chi(\mathbb{E}^2) \geq 5$ through a graph of order 1581 in 2018 [2]. Simplifying and reducing the size of this construction has been of great interest to the extent of being the topic of a Polymath project [4, 3, 10, 11].

Regarding upper bounds, there is a large number of distinct 7-colorings of the plane that avoid monochromatic pairs at unit distance, the first of which (using a tiling of the plane with congruent regular hexagons) was already observed back in 1950 by Isbell [9, 17]. This upper bound of $\chi(\mathbb{E}^2) \leq 7$ has remained unchanged since and many variants of the original question have therefore been proposed in the hopes of shedding some light on why this problem has proven so stubborn. To state one such variant, we say that an n-coloring of the plane has *coloring type* $(d_1, \ldots, d_n)$ if color i does not realize distance d_i [14, 15]. This gives a measurement of how close this coloring is to achieving the original goal and can be seen as a defining a natural 'off-diagonal' variant of the original problem. Finding a coloring of type $(1, 1, 1, 1, 1, 1)$ would obviously improve the upper bound of $\chi(\mathbb{E}^2)$ to 6.

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Stechkin found a coloring of type $(1,1,1,1,1/2,1/2)$, which was published by Raiskii in 1970 [12], and Woodall found a coloring of type $(1,1,1,1/\sqrt{3},1/\sqrt{3},1/\sqrt{12})$ in 1973 [18]. The first six-coloring to feature a non-unit distance in only one color has type $(1,1,1,1,1,1/\sqrt{5})$ and was found by Soifer in 1991 [15]. Hoffman and Soifer also found a coloring of type $(1,1,1,1,1,\sqrt{2}-1)$ in 1993 [5, 6]. Both of these constructions are in fact part of a family that realizes $(1,1,1,1,1,d)$ for any $1/\sqrt{5} \leq d \leq \sqrt{2}-1$ [6, 16, 17], leading Soifer [13] to pose the "still open and extremely difficult" [9] problem of determining the *continuum of six colorings* X_6, that is the set of all d for which there exists a six-coloring of the plane of type $(1,1,1,1,1,d)$. To the best of our knowledge, no improvements have been suggested in the last 30 years.

We propose two novel six-colorings of the plane, one parameterized by d and the other fixed, that together significantly expand the range of d known to be in X_6. The first is a valid coloring of type $(1,1,1,1,1,d)$ as long as $0.354 \leq d \leq 0.553$ and the second covers the range of $0.418 \leq d \leq 0.657$.

Theorem 1. *X_6 contains the closed interval* $[0.354, 0.657]$.

It should be noted that both constructions were derived by formalizing colorings that were suggested by a custom machine learning approach in which a Neural Network was trained to represent a coloring of a specified type or range of types. We will briefly touch upon this in Section 4 and otherwise go into more detail about this approach and potential other applications in a separate publication. This work is intended to give a formal justification of Theorem 1, with the first coloring being explored in Section 2 and the second in Section 3.

2 A construction for $0.354 \leq d \leq 0.553$

The first constructions is made up of four different polytopal shapes, a detailed description of which is given in the appendix. The equidiagonal pentagon and the equilateral triangle respectively described Figure 3 and Figure 4 together are be colored with the sixth color (red) in which we are avoiding points at distance d. The octagons described in Figure 5 receive three of the other five colors (orange, green, and blue) and the hexagons described in Figure 6 receive the remaining two (yellow and turquoise). All shapes are uniquely parameterized by the choice of d, with the exception of the pentagon, which has an additional degree of freedom in the form of α_1. We will later determine the range of valid α_1 depending on d numerically and see that this additional variable can be fixed by linearly interpolating between two extremal values (though other options can also be valid depending on d).

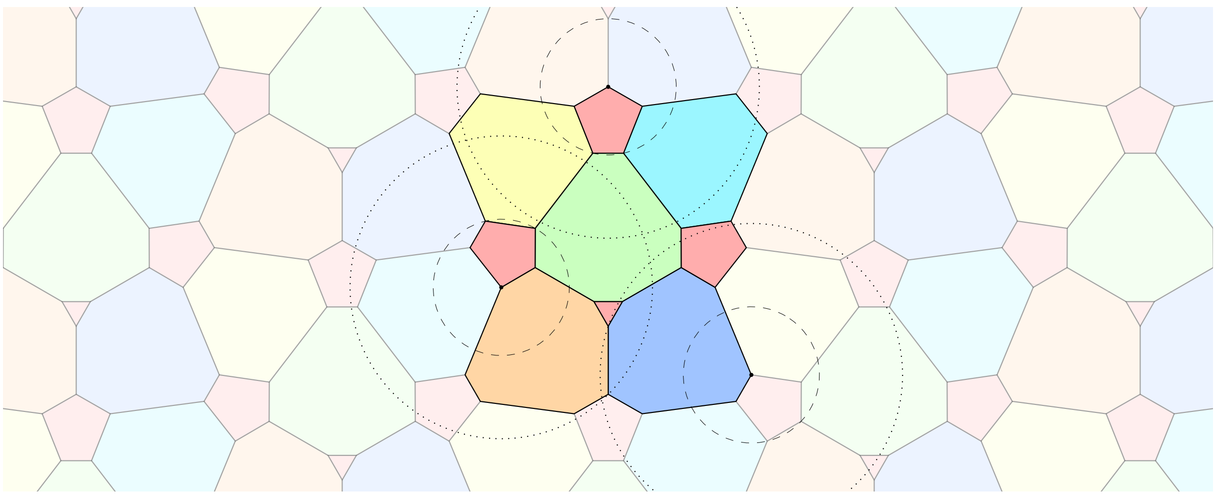

Figure 1: Illustration of the first coloring with circles at unit distance (dotted) and distance d (dashed) highlighted at three critical points.

A copy of three pentagons, one triangle, three octagons and two hexagons together form the building block of the first coloring.Note that the triangle disappears as d approaches the upper end of the valid spectrum. Looking at the overall construction in Figure 1, it is visually clear that the only conditions that are at risk making this construction invalid are given be the following set of constraints, where the variables are defined alongside the corresponding shape in the appendix:

$$s_4 \leq d \qquad (1) \qquad\qquad w_2 \leq 1 \qquad (4)$$
$$s_5 \geq d \qquad (2) \qquad\qquad w_3 \leq 1 \qquad (5)$$
$$w_1 \leq 1 \qquad (3) \qquad\qquad h_1 + h_3 + d \geq 1 \qquad (6)$$

Unfortunately we were unable to derive a closed form expression for the range of d for which a valid choice of α_1 can be found. However, it is easy to numerically verify that for $d \in [0.354, 0.553]$ such a choice can be made. Furthermore, by linearly interpolating between the two extreme points, that is by choosing $\alpha_1 = 113.7 + (d - 0.354)\,14.11/0.299$, we can remove the additional degree of freedom in the definition of the pentagon. Finally, we note that there is again always an appropriate choice for the color on the boundaries between the shapes.

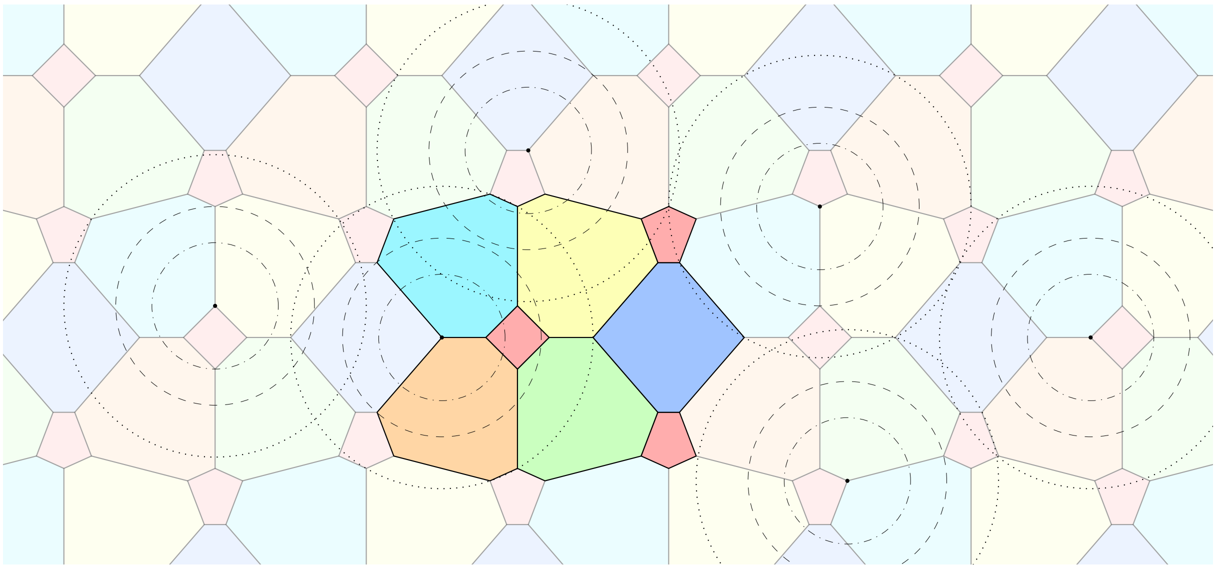

Figure 2: Illustration of the second coloring with circles at unit distance (dotted), and distance $d_{\max}$ (dashed), and distance distance $d_{\min}$ (dash-dotted) highlighted at six critical points.

3 A construction for $0.418 \leq d \leq 0.657$

Let $d_{\max}$ be the real root of $d^4 + 5\sqrt{3}d^3 + 18d^2 - 3\sqrt{3}d - 7 = 0$ closest to 0.65 and $d_{\min} = \sqrt{3} - 2\,d_{\max}$. Note that a closed form for $d_{\max}$ is given by

$$\begin{aligned} d_{\max} = &-(5\sqrt{3})/4 + 1/2\left(27/4 + 1/3\,(7290 - 15\sqrt{1821})^{1/3} + (5\,(486 + \sqrt{1821}))^{1/3}/3^{2/3}\right)^{1/2} \\ &+ 1/2\,\Big(27/2 - 1/3\,(7290 - 15\sqrt{1821})^{1/3} - (5(486 + \sqrt{1821}))^{1/3}/3^{2/3} \\ &+ 9/4\,\big(3/(27/4 + 1/3\,(7290 - 15\sqrt{1821})^{1/3} + (5(486 + \sqrt{1821}))^{1/3}/3^{2/3}))\big)^{1/2}\Big)^{1/2}. \end{aligned}$$

We can easily verify numerically that $d_{\min} \leq 0.418 \leq d \leq 0.657 \leq d_{\max}$ and the second construction will in fact be valid for any $d \in [d_{\min}, d_{\max}]$. It is again made up of four different polytopal shapes, a detailed description of which is given in the appendix. The pentagon and square described in Figure 7 together are colored with the sixth color (red) in which we are avoiding points at distance d. The heptagon described in Figure 8 receives four of the other five colors (orange, green, yellow, and turquoise) while hexagon described in Figure 8 receives the last remaining color (blue). A copy of two pentagons, one square, four heptagons and one hexagon together form the building block of the second coloring, which is illustrated in Figure 2.

4 Discussion and Outlook

We conclude by noting that there was a significant technical component to these new constructions. We developed a custom machine learning approach in which we had a Neural Network represent a (probabilistic) six-coloring of the plane. The parameters of the network were update according to a batched form of the loss given by the probabilistic likelihood that two points at unit distance (or at distance d) are monochromatic with the right color(s). The resulting output was detailed enough to inspire the above constructions, though formally describing them and verifying their correctness still required a fair amount of manual effort.

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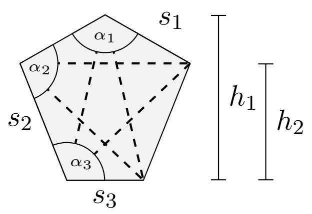
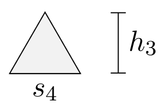
Appendix

A Building blocks of the first coloring



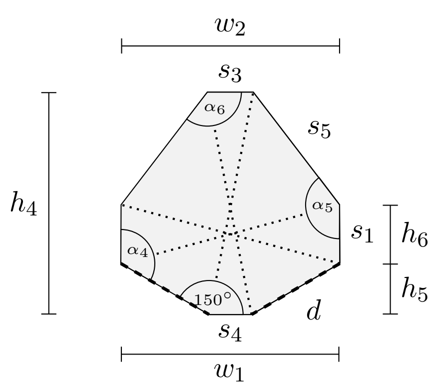
$$
\begin{aligned}
s_1 &= d/2\ \csc(\alpha_1/2)\\
t_1 &= 2\arccos\big(\csc(\alpha_1/2)/4\big) - \alpha_1\\
s_3 &= 2d\ \sin(t_1/2)\\
h_1 &= d\ \cos(t_1/2)\\
h_2 &= h_1 - (d/2)\ \cot(\alpha_1/2)\\
s_2 &= \sqrt{h_2^2 + (d-s_3)^2/4}\\
\alpha_2 &= 90^\circ - \alpha_1/2 + \arcsin(h_2/s_2)\\
\alpha_3 &= 270^\circ - \alpha_1/2 - \alpha_2
\end{aligned}
$$

Figure 3: An equidiagonal pentagon with each diagonal of length d, highlighted by dashed lines, used for the red color avoiding points at distance d in the first coloring.



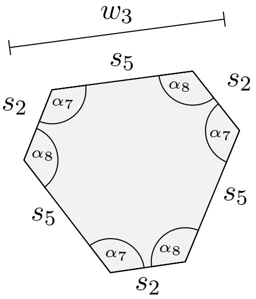
$$
\begin{aligned}
t_2 &= \big(\sqrt{1-(s_1\ \sin(30^\circ+\alpha_1/2))^2} - s_1\ \cos(30^\circ+\alpha_1/2)\big)/\sqrt{3}\\
s_4 &= \sqrt{3}\ \max(t_2-d,0)\\
h_3 &= 3/2\ \max(t_2-d,0)
\end{aligned}
$$

Figure 4: An equilateral triangle, used for the red color avoiding points at distance d in the first coloring.



$$
\begin{aligned}
w_1 &= \sqrt{3}\,t_2\\
t_3 &= 180^\circ - \arccos\big((1-w_1^2-s_1^2)/(-2w_1s_1)\big)\\
w_2 &= s_1\cos(t_3) + \sqrt{1-s_1^2\sin(t_3)^2}\\
h_4 &= \sqrt{1-(s_4+s_3)^2/4}\\
h_5 &= \sqrt{t_2^2 - w_1^2/4} - h_3 + \max(t_2-d,0)\\
h_6 &= \sqrt{s_1^2-(w_1-w_2)^2/4}\\
s_5 &= \sqrt{h_7^2+(w_2-s_3)^2/4}\\
\alpha_4 &= 180^\circ - \alpha_1/2\\
\alpha_5 &= \arctan\big(2\,h_7/(w_2-s_3)\big) + t_3\\
\alpha_6 &= 390^\circ - \alpha_4 - \alpha_5
\end{aligned}
$$

Figure 5: An axisymmetric octagon in which four of the diagonals have unit length, highlighted by dotted lines, and two of the sides have length d, highlighted by dashed lines. Used for the orange, green and blue color avoiding points at unit distance in the first coloring.



$$\alpha_7 = 360^\circ - \alpha_2 - \alpha_5$$
$$\alpha_8 = 360^\circ - \alpha_3 - \alpha_6 = 240^\circ - \alpha_7$$
$$t_4 = \sqrt{s_2^2 + s_5^2 - 2s_2s_5\cos(\alpha_7)}$$
$$t_5 = \arcsin(s_5\sin(\alpha_7)/t_4)$$
$$w_3 = \sqrt{t_4^2 + s_2^2 + 2t_4s_2\cos(\alpha_7 + \alpha_8 + t_5)}$$

Figure 6: A hexagon with two angles and two side lengths. Used for the yellow and turqouise color avoiding points at unit distance in the first coloring. Note that it is in general *not* axisymmetric.

B Building blocks of the second coloring

Figure 7: An axisymmetric pentagon and a square together are used for the red color avoiding points at distance d in the second coloring. s_2 and s_3 are implicitly defined in Figure 8.

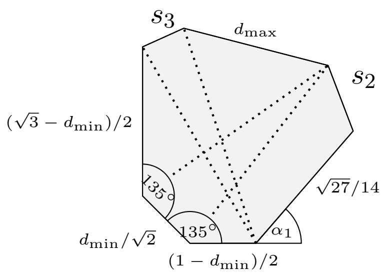
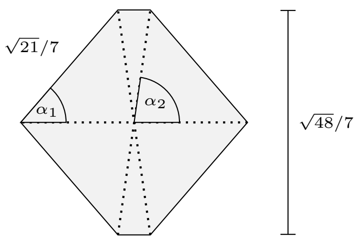


Figure 8: A heptagon in which four of the diagonals have unit length, highlighted by dotted lines. Used for the orange, green, yellow, and turquoise color avoiding points at unit distance in the second coloring. We do not give a closed form solution for s_2 and s_3 but note that they are well defined. The angle α_1 is defined in Figure 9.



$$\alpha_1 = 45^\circ + \arccos(47/49)/4$$
$$\alpha_2 = 90^\circ - \arccos(47/49)/2$$

Figure 9: A centrosymmetric hexagon in which three of the diagonals have unit length, highlighted by dotted lines. Used for the blue color avoiding points at unit distance in the second coloring.