Rainbow connectivity of multilayered random geometric graphs

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Abstract

An edge-colored multigraph G is rainbow connected if every pair of vertices is joined by at least one rainbow path, i.e., a path where no two edges are of the same color. In the context of multilayered networks, we introduce the notion of multilayered random geometric graphs, from $h \geq 2$ independent random geometric graphs $G(n,r)$ on the unit square. We define an edge-coloring by coloring the edges according to the copy of $G(n,r)$ they belong to and study the rainbow connectivity of the resulting edge-colored multigraph. We show that $r(n) = \left(\frac{\ln n}{n^{h-1}}\right)^{1/2h}$, is a threshold of the radius for the property of being rainbow connected. This complements the known analogous results for the multilayered graphs defined on the Erdős–Rényi random model.

1 Introduction

Complex networks are used to simulate large-scale real-world systems, which may consist of various interconnected sub-networks or topologies. For instance, this could involve different transportation systems and coordinating schedules between them, modeling interactions across different topologies of the network. Barrat et al. [1] proposed a new network model to represent the emerging large network systems, which include coexisting interacting different topologies. Those network models are known as *layered complex networks, multiplex networks* or as *multilayered networks.* In a multilayered network, each type of interaction of the agents gets its own layer, like a social network having a different layer for each relationship, such as friendship or professional connections [6]. Recently, there's been a lot of interest in adapting tools used in the analysis for single-layer networks to the study of multilayered ones, both in deterministic and random models [2]. In the present work, we explore thresholds for the *rainbow connectivity of the multilayered random geometric graphs.*

A *random geometric graph* (RGG), $G(n,r)$, where $r = r(n)$ on the unit square $I = [0,1]^2$ is defined as follows: Given n vertices and a radii $r(n) \in [0, \sqrt{2}]$, n vertices are sprinkled independently and uniformly at random (u.a.r.) in the unit square $I = [0,1]^2$. Two vertices are adjacent if and only if their Euclidean distance is less than or equal to $r(n)$.

Random geometric graphs provide a natural framework for the design and analysis of wireless networks. For further information on random geometric graphs, one may refer to Penrose [10] or to the more recent survey by Walters [12]. Random geometric graphs exhibit a sharp threshold behavior with respect to connectivity [7]: As the value of r increases, there is a critical threshold value r_c such that

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when $r < r_c$, the graph is typically disconnected, while for $r > r_c$, the graph is typically connected. The threshold for connectivity of $G(n,r)$ is $r_c \sim \sqrt{\frac{\ln n}{\pi n}}$. Notice r_c is also a threshold for the disappearance of isolated vertices in $G(n,r)$.

For any random geometric graph, $G(n,r)$, the expected degree $|N_{G(n,r)}(v)|$ is w.h.p.[1] $n\pi r^2, \forall v \in V(G)$. Equivalently the expected degree is concentrated around its mean. Regarding the diameter, $diam(G)$, of a random geometric graph $G(n,r)$, Díaz et al. [5] showed that if $r = \Omega(r_c)$ then $diam(G) = (1+o(1))\frac{\sqrt{2}}{r}$.

We now introduce a general definition for the random model of edge colored multigraphs obtained by the superposition of a collection of random geometric graphs on the same set of vertices. Formally, a *multilayered geometric graph* $G(n,r,h,b)$ is defined by three parameters, n the number of nodes, r the radii of connectivity, and h the number of layers, together with a position assignment $b : [n] \to \underbrace{[0,1]^2 \times \cdots \times [0,1]^2}_{h}$. For $i \in [n]$, we denote $b(i) = (b_1^i, \ldots, b_h^i)$, where $b_k^i \in [0,1]^2$. The multigraph $G(n,r,h,b)$ has vertex set $[n]$ and an edge (i,j) with color k, $1 \le k \le h$, if the Euclidean distance between b_k^i and b_k^j is at most r. Note that, for $k \in [h]$, r and the positions $(b_k^i)_n$, a geometric graph $G_k(n,r)$ is defined by the edges with color k. Thus, $G(n,r,h,b)$ can be seen as the colored union of h geometric graphs, all with the same vertex set and radius. Observe that $G(n,r,h,b)$ is defined on I^{2h}. We refer to $G_k(n,r)$ as the k-th layer of $G(n,r,h,b)$.

A *multilayered random geometric graph* $G(n,r,h)$ is obtained when the position assignment b of the vertices is selected independently, for each vertex and layer, uniformly at random in $[0,1]^2$. Thus, the k-th layer is an RGG. This definition is given for dimension two and it can be extended to points in a multidimensional space by redefining the scope of the position function.

Given an edge–colored graph G, we say G is *rainbow connected* if, between any pair of vertices $u, v \in V(G)$, there is a path with edges of pairwise distinct colors. Chartrand et al [4] introduced the study of the rainbow connectivity of graphs as a strong property to secure strong connectivity in graphs and networks. Since then, variants of rainbow connectivity have been applied to different deterministic models of graphs, see for ex. the survey of Li et al. [8] for further details on the extension of rainbow connectivity to other graph models.

The study of rainbow connectivity has been addressed in the context of multilayered binomial random graphs by Bradshaw and Mohar [3]. The authors give sharp concentration results on three values on the number h of layers needed to ensure rainbow connectivity of the resulting multilayered binomial random graph $G(n,p)$ with appropriate values of p. The results have been extended by Shang [11] to ensure rainbow connectivity k in the same model, namely, the existence of k internally disjoint rainbow paths joining every pair of vertices in the multilayered graph.

In this paper, we are interested in studying the rainbow connectivity of a multilayered random geometric graph $G(n,r,h)$. In particular, for every fixed h, we are interested in the minimum value of r (as a function of n) such that w.h.p. the multilayered random geometric graph $G(n,r,h)$ is rainbow connected. Dually, for fixed values of r we want to determine the minimum number of layers h such that $G(n,r,h)$ is rainbow connected. The latter parameter can be defined as the rainbow connectivity of the multilayered random geometric graph.

Main results: Our main results are lower and upper bounds of the value of r, to asymptotically assure that w.h.p. $G(n,r,h)$, do have or do not have the property of being rainbow connected.

Theorem 1. *Let $h \ge 2$ be an integer and let $G = G(n,h,r)$ be an h–layered random geometric graph. Then, if*

$$r(n) \ge \left(\frac{\ln n}{n^{h-1}}\right)^{1/2h},$$

[1] w.h.p. means with high probability, i.e. with probability tending to 1 as $n \to \infty$.

then w.h.p. G is rainbow connected.

Moreover, there is a constant $0 < c \leq 1$ such that, if

$$r(n) < c\left(\frac{\ln n}{n^{h-1}}\right)^{1/2h},$$

then w.h.p. G is not rainbow connected.

Notice that Theorem 1 can be re-stated as a threshold of h for the rainbow connectivity of multilayered geometric random graph G.

Corollary 2. *Let $r = r(n)$ with $r(n) = o(1)$. Set*

$$h_0 = \left\lceil \frac{\log n + \log\log n}{\log nr^2} \right\rceil.$$

The multilayered random geometric graph $G(n, r, h)$ is w.h.p. rainbow connected if $h \leq h_0$, while if $h > h_0$ it is w.h.p. not rainbow connected.

2 Rainbow Connectivity of Two-layered Random Geometric Graphs

The proof of Theorem 1 requires a special argument for the case $h = 2$. We give below the proof of this case which also illustrates the techniques for general $h > 2$.

Proposition 3. *Let $G(n, r, 2)$ be a two–layered random geometric graph. If*

$$r(n) \geq \left(\frac{\ln n}{n}\right)^{1/4},$$

then G is w.h.p. rainbow connected.

Moreover, there is a positive constant $c > 0$ such that, if

$$r(n) \leq c\left(\frac{\ln n}{n}\right)^{1/4},$$

then w.h.p. G is not rainbow connected.

Proof. Denote by $G_1(n, r)$ and $G_2(n, r)$ the two layers of G, with the value of $r = r(n)$ given in the statement of the proposition. For each pair $v_i, v_j \in V$, let X_{v_i,v_j} denote the indicator random variable

$$X_{v_i,v_j} = \begin{cases} 1 & \text{if there is not a rainbow path between } v_i \text{ and } v_j \text{ in } G\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let v_k be different from v_i and v_j. Let A_{v_k} be the event that v_k is joined to v_i in $G_1(n, r)$ and to v_j in $G_2(n, r)$ or vice versa, namely,

$$A_{v_k} = \{\{v_i \in \mathcal{B}_1(v_k)\} \cap \{v_j \in \mathcal{B}_2(v_k)\}\} \cup \{\{v_j \in \mathcal{B}_1(v_k)\} \cap \{v_i \in \mathcal{B}_2(v_k)\}\}\,,$$

where $\mathcal{B}_i(v)$ denotes the set of neighbours of v in G_i, $i = 1, 2$. By taking into account the boundary effects on the unit square, we have $\Pr(v_i \in \mathcal{B}(v_j)) = \pi r^2 + o(r^2)$. We have,

$$(\pi r^2 + o(r^2))^2 \leq \Pr(A_{v_k}) \leq 2(\pi r^2 + o(r^2))^2\,.$$

Let $A_{v_i v_j}$ denote the event that v_i and v_j are joined by an edge either in $G_1(n, r)$ or in $G_2(n, r)$, that is

$$A_{v_i,v_j} = \{v_i \in \mathcal{B}_1(v_j)\} \cup \{v_i \in \mathcal{B}_2(v_j)\},$$

so that

$$\Pr(A_{v_i,v_j}) = 2\pi r^2 + o(r^2).$$

For given v_i and v_j, the event that they are joined by a rainbow path in G is $(\cup_{k\neq i,j} A_{v_k}) \cup A_{v_i,v_j}$. Therefore, since A_{v_k} and A_{v_i,v_j} are independent, for every sufficient large n we have

$$\begin{aligned}\mathbb{E}(X_{v_i,v_j}) &= \Pr(\overline{(\cup_{k\neq i,j} A_{v_k}) \cup (A_{v_i,v_j})}) = \Pr((\cap_{k\neq i,j} \overline{A_{v_k}}) \cap \overline{(A_{v_i,v_j})}) \\ &\leq (1-(\pi r^2)^2 + o(r^2))^{n-2} \cdot (1-2\pi r^2 + o(r^2)) \\ &\leq (1-(\pi r^2)^2 + o(r^2))^n .\end{aligned}$$

Let X be a random variable counting the number of pairs $\{v_i, v_j\}$ that are not joined by a rainbow path in G. Then $X = \sum_{i<j} X_{v_i,v_j}$ and, by plugging in the inequality for $r(n)$,

$$\begin{aligned}\mathbb{E}(X) = \sum_{i<j} \mathbb{E}(X_{v_i,v_j}) &\leq \binom{n}{2}(1-(\pi r^2)^2 + o(r^2))^n \\ &\leq e^{2\log n}\left(1-\pi^2\frac{\log n}{n} + o\left(\frac{\log n}{n}\right)\right)^n \leq e^{(2-\pi^2)\log n + o(\log n)}\end{aligned}$$

By Markov's inequality, it follows that $\Pr(X \geq 1) \leq \mathbb{E}(X) \to 0$, as $n \to \infty$. It follows that w.h.p. G is rainbow connected, which proves the first part of the statement.

For the second part, let $r(n) \leq c(\log n/n)^{1/4}$ for some positive small constant c to be specified later. By using the upper bounds on the probabilities of the events A_{v_k} and A_{v_i,v_j},

$$\mathbb{E}(X_{v_i,v_j}) \geq (1-2(\pi r^2 + o(r^2))^2)^{n-2}(1-2\pi r^2 + o(r^2)) \geq \left(1-2c^4\pi^2\frac{\ln n}{n}\right)^{n-1} .$$

$$\mathbb{E}(X_{v_i,v_j}) \geq (1-2(\pi r^2 + o(r^2))^2)^{n-2} \geq \left(1-2c^4\pi^2\frac{\ln n}{n} + o\left(\frac{\log n}{n}\right)\right)^{n-2}$$

Let $X_{v_i} = \sum_{j\neq i} X_{v_i,v_j}$ denote the number of vertices v_j not joined with v_i by a rainbow path in G. We have, with $c' = 2c^4\pi^2$,

$$\mathbb{E}(X_{v_i}) \geq (n-2)\left(1-c'\frac{\ln(n-1)}{n-1} + o\left(\frac{\log n}{n}\right)\right)^{n-2} \sim e^{(1-c')\ln n} = n^{1-c'}.$$

By choosing $c < (2/\pi^2)^{1/4}$ we have $c' < 1$, so that $\mathbb{E}(X_i) \to \infty$ with $n \to \infty$. Since X_{v_i} is a sum of independent random variables, by Chernoff inequality we have $\Pr(X_{v_i} = 0) \leq e^{-n^{1-c''}/2}$ for each $1 > c'' > c'$. It follows that G is w.h.p. not rainbow connected. □

3 Proof of Theorem 1

The proof of Theorem 1, for $h > 2$, is sketched below.

A key property of multilayered random geometric graphs is their local expanding properties.

Lemma 4. *Let $h > 2$ be fixed and let $G = G(n, r, h)$ be a multilayered random geometric graph. Let $u \in V(G)$ a fixed vertex and denote by $N_j(u)$ the set of vertices reached from u by rainbow paths of length j starting at u, the i–th edge along the path colored i. Let $M = nr^2$. Then, for $1 \leq j \leq h-1$ we have that w.h.p.*

$$|N_j(u)| = \Theta(M^j).$$

The proof of Lemma 4 uses the fact that the probability that the size of the image of a random map $g : [m] \to [k]$ deviates from m more than a constant $a > 0$ is at most $2\exp(-2(a - m^2/2k)^2/m)$. This fact in turn follows by a direct application of the McDiarmid concentration inequalities [9].

Lemma 4 provides the existence of rainbow paths from a given vertex to all vertices in the graph.

Proposition 5. *Let $h > 2$ be fixed and let $G = G(n, h, r)$ be an h–multilayered random geometric graph. Let $u \in V(G)$. If*

$$r \geq \left(\frac{\ln n}{n^{h-1}}\right)^{1/2h},$$

then w.h.p. there is a rainbow path from u to every other vertex in G.

Proof. Let us consider first the case that $h \geq 3$ is odd, i.e., $h = 2k + 1$, for some $k > 1$. Denote by $G_i = G_i(n, r)$ the i-thlayer of G. For $I \subseteq [h]$, we denote by $G_I(n, r)$ the layered graph formed by the layers included in I. For a pair i, j of distinct vertices in $V(G)$ and a permutation σ of $\{1, 2, 3, \ldots, h\}$, let $P(i, j; \sigma)$ denote the set of rainbow paths of length h joining i and j with the first edge in $G_{\sigma(1)}$ and the last one in $G_{\sigma(h)}$. For a permutation σ, let $I_1(\sigma) = \{\sigma(1), \ldots, \sigma(k)\}$

Let $A = N_{k,\sigma}(i)$ be the set of vertices reached from i by rainbow paths of length k starting at j following the color order determined by σ. Let $B = N_{k,\sigma}(j)$ be the set of vertices reached from j by rainbow paths of length k starting at j following the color order determined by following σ in reversed order with the k-th edge along the path colored $k+2$. From Lemma 4, $|A|, |B| = \Theta((nr^2)^k) = \Theta(n^k r^{2k})$

Let $X_{i,j}$ denote the number of rainbow paths of length h joining i and j with the first edge in $G_{\sigma(1)}$, the second edge in $G_{\sigma(2)}$ and so on. For a pair $(k, k') \in A \times B$ with $k \neq k'$, let $Y_{k,k'}$ be the indicator function that k and k' are neighbours in $G_{\sigma(k+1)}$. We have $\mathbb{E}(Y_{k,k'}) = \pi r^2$, the probability that the vertices k' and k are adjacent in $G_{\sigma(k+1)}$. Then,

$$X_{ij} = \sum_{k,k'} Y_{k,k'},$$

where the sum runs through all pairs $(k, k') \in A \times B$ with $k \neq k'$. We observe that the variables $Y_{k,k'}$ are independent. When the pairs $(k, k'), (l, l')$ are disjoint it is clear that $Y_{k,k'}, Y_{l,l'}$ are independent. When $k = l$, say, then $\Pr(Y_{k,k'} = 1, Y_{k,l'} = 1)$ is the probability that k' and l' are both adjacent to k, which is the product $\Pr(Y_{k,k'} = 1) \Pr(Y_{k,l'} = 1)$.

Let us fix $r(n) \geq \left(\frac{\ln n}{n^{h-1}}\right)^{1/2h}$. Note that $N_{h-1}(u) \ll n$, so each $(h-1)$–layered subgrah of G is not w.h.p. rainbow connected. Then it follows that w.h.p. the sets A and B, for $i \neq j$ not connected by a rainbow path of length $h - 1$ are disjoint. In this case, the events $Y_{k,k'}$ are independent, therefore

$$\begin{aligned}\Pr(X_{i,j} = 0) = \Pr(\cap_{k,k'}\{Y_{k,k'} = 0\}) &= \prod_{k,k'} \Pr(Y_{k,k'} = 0) \\ &= (1 - \pi r^2)^{(n^k r^{2k})^2} \leq e^{-\pi n^{2k} r^{4k+2}}.\end{aligned}$$

By using the union bound on all pairs i, j and the lower bound on r,

$$\Pr(\cap_{i,j}\{X_{ij} \geq 1\}) = 1 - \Pr(\cup_{i,j} X_{i,j} = 0) \geq 1 - n^2 e^{\pi n^{2k} r^{4k+2}},$$

As $k = (h - 1)/2$, by the lower bound on r,

$$n^{2k} r^{4k+2} = n^{h-1} r^{2h} \geq (\log n),$$

Therefore, the last term in the bound on $\Pr(\cap_{i,j}\{X_{ij} \geq 1\})$ is $o(1)$ as $n \to \infty$. Hence w.h.p. all pairs i, j are connected by a rainbow path of length h.

For even h, the result is obtained by an extension of the argument used for $h = 2$ in Proposition 3. □

For the lower bound on $r(n)$, an application of the second moment method as the one given in Proposition 3 for the case $h = 2$ can be extended to $h > 2$.

4 Conclusions

The main purpose of this paper is to identify the threshold for the radius to get a rainbow-connected multilayered random geometric graph, as obtained in Theorem 1. As mentioned in the Introduction, the analogous problem of determining the threshold for h so that the multilayered binomial random graph is rainbow connected was addressed by Bradshaw and Mohar [3].

We believe that the model of multilayered random geometric graphs is very appealing and leads to a host of interesting problems. One may think of a dynamic setting where n individuals perform random walks within the cube and communicate with the close neighbors at discrete times $t_1 < t_2 < \cdots < t_h$. The rainbow connectivity in this setting measures the number of instants needed so that every individual can communicate with each of the other ones. A natural immediate extension is to address the threshold to get rainbow connectivity k, as achieved in the case of multilayered binomial random graphs by Shang [11].

There is a vast literature addressing rainbow problems in random graph models, and this paper is meant to open the path to these problems in the context of multilayered random geometric graphs. It would also be interesting to find asymptotic estimates on r such that h copies produce a rainbow clique of size $\sqrt{h}$.

We observe that, for large h, the threshold of r for rainbow connectivity approaches the connectivity threshold of random geometric graphs. The arguments in the proof, however, apply only for constant h. For h growing with n, the correlation between distinct edges in our model decreases and the model gets closer to the random binomial graph, where the results are expected to behave differently and the geometric aspects of the model become irrelevant.

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