The Four-Color Ramsey Multiplicity of Triangles*

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Abstract

We study a generalization of a famous result of Goodman and establish that asymptotically at least a $1/256$ fraction of all triangles needs to be monochromatic in any four-coloring of the edges of a complete graph. We also show that any large enough extremal construction must be based on a blow-up of one of the two $R(3,3,3)$ Ramsey-colorings of K_{16}. This result is obtained through an efficient flag algebra formulation by exploiting problem-specific combinatorial symmetries that also allows us to study some related problems.

1 Introduction

In 1959, Goodman [17] established precisely how few monochromatic triangles any two-edge-coloring of the complete graph on n vertices can contain, implying that asymptotically at least $1/4$ of all triangles need to be monochromatic as n tends to infinity. Subsequently, in [18], he also asked for an answer to the natural generalization of this problem to more than two colors.[1] It took over 50 years and the advent of flag algebras for even the case of three colors to be settled: Cummings et al. [7] showed that asymptotically at least a $1/25$ fraction of all triangles need to be monochromatic in any three-edge-coloring of K_n. For n large enough they also precisely characterize the set of extremal constructions, showing that the problem is closely linked to the Ramsey Number $R(3,3)=6$ as previously noted by Fox [12, Theorem 5.2]. The purpose of this paper is to study the next iteration of this problem, in particular establishing an answer in the affirmative to Question 4 in [7] for the case of four colors.

Theorem 1. *Asymptotically at least a $1/256$ fraction of all triangles are monochromatic in any four-edge-coloring of K_n and any sufficiently large extremal coloring must be based on one of the two $R(3,3,3)$ Ramsey-colorings of K_{16}.*

The proof of this result relies on the flag algebra framework of Razborov [32, 6]. This allows one to apply a formalized double counting and Cauchy-Schwarz-type argument to obtain bounds for classic problems in Turán and Ramsey theory by solving a concrete semidefinite programming (SDP) formulation. Broadly speaking, the larger this formulation, the better the derived bound becomes.

The major hurdle in establishing Theorem 1 therefore consisted of deriving an efficient formulation by identifying and exploiting combinatorial symmetries through a parameter-dependent notion of automorphisms. The resulting proof likely constitutes the largest *exact* flag algebra calculation done to date. The methods developed to derive it strengthen the previous approach of modifying the underlying notion of isomorphism and generalize Razborov's invariant-anti-invariant decomposition [33]. They

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[1]He in fact calls the three-color version of this question "an old and difficult problem" and raises the question of more than three colors in Section 6 of [18]. The precise origin of this problem is unclear.

are applicable whenever the object we are minimizing has previously ignored symmetries and we hope that they will therefore find further applications. Accompanying these computational improvements, we also give an extension of the stability argument previously developed for the three-color case in [7]. We generalize it to the case of an arbitrary number of colors and establish a strong link between the problem of determining the Ramsey number and the Ramsey multiplicity problem.

2 The Ramsey Multiplicity Problem

We are studying the family of c-colorings of the edges between a finite number of vertices, that is maps $G : \big\{\{u, v\} \mid u, v \in V,\, u \neq v\big\} \to [c] = \{1, \ldots, c\}$ where V is any finite set, but we will use common graph notation throughout. Let $\mathcal{G}^{(c)}$ denote the set of all such colorings and $\mathcal{G}^{(c)}_n$ the set of all colorings of order n. Given colorings $H \in \mathcal{G}^{(c)}_k$ and $G \in \mathcal{G}^{(c)}_n$, we write $p(H; G) = |\{S \subseteq V(G) \mid G[S] \simeq H\}|/\binom{n}{k}$ for the *density* of H in G. Note that $p(H; G) = 0$ if $n < k$. Denoting the monochromatic coloring of the edges between vertices in $[t]$ with color $i \in [c]$ by K^i_t, a multi-color version of Ramsey's theorem states that for any $t_1, \ldots, t_c \in \mathbb{N}$ the number $R(t_1, \ldots, t_c) = \min\big(\{n \in \mathbb{N} \mid \{G \in \mathcal{G}^{(c)}_n \mid p(K^1_t; G) + \ldots + p(K^c_t; G) = 0\} = \emptyset\}\big)$ is in fact finite. For the diagonal case, where $t_1 = \ldots = t_c$, we write $R_c(t) = R(t, \ldots, t)$. The study of the parameter

$$m_c(t; n) = \min_{G \in \mathcal{G}^{(c)}_n} p(K^1_t; G) + \ldots + p(K^c_t; G)$$

is known as the *Ramsey multiplicity problem* for cliques. A simple double-counting argument establishes that $m_c(t; n)$ is monotonically increasing, so that the limit $m_c(t) = \lim_{n\to\infty} m_c(t; n)$ is well defined and satisfies $m_c(t) \geq m_c(t; n)$ for any $n \in \mathbb{N}$. Note that $m_c(t; n) > 0$ as long as $n \geq R_c(t)$ and therefore $m_c(t) > 0$ by Ramsey's theorem.

Concerning upper bounds for $m_c(t)$, coloring the edges uniformly at random with the c colors establishes that

$$m_c(t) \leq c^{1-\binom{t}{2}}. \tag{1}$$

Another way to obtain an upper bound is by blowing up a coloring of the edges of a *looped* complete graph, that is a map $C : \big\{\{u, v\} \mid u, v \in V\big\} \to [c]$. We use the same notation concerning the vertex and edge set as we did for unlooped colorings and write $\mathcal{L}^{(c)}$ for the set of all such colorings as well as $\mathcal{L}^{(c)}_n$ for colorings of order n. A coloring $H \in \mathcal{G}^{(c)}$ *embeds* into a given $C \in \mathcal{L}^{(c)}_k$, if there exists a (not necessarily injective) map $\varphi : V(H) \to V(C)$ satisfying $H(\{u, v\}) = C(\{\varphi(u), \varphi(v)\})$ for all $u, v \in V(H)$. We now let $\mathcal{B}(C) = \{H \in \mathcal{G}^{(c)} \mid H$ embeds into $C\}$ denote the *family of blow-up colorings* of C. Note that $\mathcal{B}(C)$ contains graphs of arbitrarily large order. Letting

$$\hat{p}(H; C) = |\{\varphi \text{ embeds } H \text{ into } C\}| \,/\, v(C)^{v(H)}$$

denote the *embedding density*, we have the following result

Lemma 2. *Given any $C \in \mathcal{L}^{(c)}_k$, we have $m_c(t) \leq \hat{p}(K^1_t; C) + \ldots + \hat{p}(K^c_t; C)$.*

In our case, the most relevant candidates for colorings C are obtained by considering a Ramsey-coloring on $r = R_{c-1}(t) - 1$ vertices avoiding cliques of size t in any of the $c - 1$ colors. Coloring the loops with the additional c-th, this implies an upper bound of

$$m_c(t) \leq (R_{c-1}(t) - 1)^{1-t}, \tag{2}$$

see also Theorem 5.2 in [12]. The result of Goodman [17] implies that $m_2(3) = 1/4$. This aligns both with the probabilistic upper bound stated in Equation (1) as well as the Ramsey upper bound stated in Equation (2), where for the latter we are relying on the trivial case of Ramsey's theorem, that is $R_1(t) = R(t) = t$.

Given that the former bound dominates when t grows as long as $c = 2$, Erdős suggested [10] that the probabilistic upper bound should always be tight in this case. This was disproven by Thomason [39] for any $t \geq 4$ and a significant number of results since then have tried to either determine improved asymptotics for $m_2(t)$ or specific values of it for small t [5, 9, 11, 14, 15, 16, 20, 22, 27, 35, 38, 40, 41, 29]. The problem also links to Sidorenko's famous open conjecture and the search for a characterization of common graphs, cf. [4, 36]. As of now, even $m_2(4)$ remains open, with the best current lower and upper bounds of $0.0296 \leq m_2(4) \leq 0.03014$ respectively due to Grzesik et al. [20] as well as Parczyk et al. [29]. Note that we also obtained a slight improvement of $m_2(4) \geq 0.02961$.

For the asymptotic values, there has likewise been scarce progress, with the current best lower bound of $C^{-t^2(1+o(1))} \leq m_2(t)$ for $C \approx 2.18$ due to Conlon [5] and the best upper bound of $m_2(t) \leq 0.835 \cdot 2^{1-\binom{t}{2}}$ for $t \geq 7$ due to Jagger, Šťovíček, and Thomason [22]. Given the lack of progress on the two-color, diagonal version, there are two obvious directions to explore: the case of more colors, where $c > 2$, as well as the off-diagonal case, where $t_1 \neq t_2$.

3 Increasing the number of colors

Studying monochromatic triangles for more than two colors was, as already mentioned in the introduction, suggested by Goodman [18] and resolved for $c = 3$ by Cummings et al. [7], whose result aligns with Equation (2) since $R_2(3) = 6$. In order to state their result in its fullest strength, let $C_{R(3,3)}$ denote the coloring in $\mathcal{L}_5^{(3)}$ obtained by taking the unique Ramsey 2-coloring of a complete graph on five vertices that avoids monochromatic triangles and coloring the loops with the third color, that is $E_1(C_{R(3,3)})$ and $E_2(C_{R(3,3)})$ both are 5-cycles and $E_3(C_{R(3,3)})$ contains all five loops. Let $\mathcal{G}_{\mathrm{ex}}^{(3)} \subset \mathcal{G}^{(3)}$ now consist of all colorings that can be obtained by (i) selecting an element in $\mathcal{B}(C_{R(3,3)})$, (ii) recoloring some of the edges from the first or second color to instead use the third color without creating any additional monochromatic triangles, and (iii) applying any permutation of the colors. Note that the second step implies that the recolored edges must form a matching between any of the five 'parts', though not every such recoloring avoids additional triangles.

Theorem 3 (Cummings et al. [7])**.** *There exists an $n_0 \in \mathbb{N}$ such that any element in $\mathcal{G}_n^{(3)}$ of order $n \geq n_0$ minimizing the number of monochromatic triangles must be in $\mathcal{G}_{ex}^{(3)}$.*

The result characterizes extremal constructions for large enough n, though more recently there has been increasing interest in deriving stability results based on flag algebra calculations [30]. Let $C'_{R(3,3,3)}$ and $C''_{R(3,3,3)}$ denote the two colorings in $\mathcal{L}_{16}^{(4)}$ obtained in a similar way to the previously defined $C_{R(3,3)}$ by respectively taking the two Ramsey 3-coloring of a complete graph on 16 vertices that avoid monochromatic triangles [19, 23, 24, 31] and coloring the vertices with the fourth color. Mirroring the construction of $\mathcal{G}_{\mathrm{ex}}^{(3)}$, we let $\mathcal{G}_{\mathrm{ex}}^{(4)} \subset \mathcal{G}^{(4)}$ consist of all colorings that can be obtained by

(i) selecting an element in $\mathcal{B}(C'_{R(3,3,3)})$ or $\mathcal{B}(C''_{R(3,3,3)})$,

(ii) recoloring some of the edges from any of the first, second or third color to instead use the fourth color without creating any additional monochromatic triangles,

(iii) applying any permutation of the four colors.

Theorem 4. *There exists an $n_0 \in \mathbb{N}$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that any $G \in \mathcal{G}_n^{(4)}$ of order $n \geq n_0$ with $\sum_{i=1}^{c} p(K_3^i; G) \leq m_4(3; n) + \delta$ can be turned into an element of $\mathcal{G}_{ex}^{(4)}$ by recoloring at most $\varepsilon\binom{n}{2}$ edges.*

Note that this implies that any large enough element in $\mathcal{G}_n^{(4)}$ minimizing the number of monochromatic triangles must be in $\mathcal{G}_{\mathrm{ex}}^{(4)}$. Our results in fact show that it likewise can be obtained for the case of three colors.

4 The off-diagonal case

The second of the previously suggested directions, that is considering the off-diagonal case, has recently started to receive some attention [29, 3, 26, 21] with two competing notions of off-diagonal Ramsey multiplicity having been suggested. The first is due to Parczyk et al. [29] and is concerned with determining

$$m(t_1, \ldots, t_c; n) = \min_{G \in \mathcal{G}_n^{(c)}} p(K^1_{t_1}; G) + \ldots + p(K^c_{t_c}; G).$$

This generalizes the previously defined $m_c(t; n)$ but does not consider the inherent imbalance when for example $c = 2$ and $t_1 \ll t_2$; minimizing $p(K^1_{t_1}; G) + p(K^2_{t_2}; G)$ in this case will be equivalent to enforcing $p(K^1_{t_1}; G) = 0$ and minimizing $p(K^2_{t_2}; G)$, a related problem previously suggested by Erdős [10, 28, 8, 29]. This issue was already noted in [29] and subsequently addressed by Moss and Noel [26], who instead suggested determining

$$m_s(t_1, \ldots, t_c; n) = \min_{G \in \mathcal{G}_n^{(c)}} \max_{\substack{\lambda_1, \ldots, \lambda_c \geq 0 \\ \lambda_1 + \ldots + \lambda_c = 1}} \lambda_1 \, p(K^1_{t_1}; G) + \ldots + \lambda_c \, p(K^c_{t_c}; G).$$

We will use $m(t_1, \ldots, t_c)$ as well as $m_s(t_1, \ldots, t_c)$ to respectively denote the limits of both of these functions as n tends to infinity. Both notions generalize the previous diagonal definition and clearly $m_s(t_1, \ldots, t_c) \geq m(t_1, \ldots, t_c)$. Unsurprisingly, determining $m_s(t_1, \ldots, t_c)$ has proven much more difficult, with $m(3, 4)$ and $m(3, 5)$ having been settled in [29] and $m_s(3, 4)$ still remaining open. Here we derive the following result for the weaker of the two notions.

Proposition 5. *We have $m(3, 3, 4) = 1/125$.*

The upper bound follows immediately by generalizing Equation (2) to the off-diagonal case, that is by noting that

$$m(t_1, \ldots, t_c) \leq (R(t_1, \ldots, t_{c-1}) - 1)^{1-t_c} \tag{3}$$

and inserting $R(3, 3) = 6$. The lower bound was derived using the same improvements to the flag algebra calculus that we developed to derive our main result.

5 Discussion and Outlook

The proposed computational improvements were crucial in order to derive a certificate for the upper bound and stability statement in Theorem 4. They are applicable whenever the problem studied exhibits symmetries with respects to the colors, with the reduction of the number of constraints essentially factorial in the number of colors. We therefore hope that they find further use for other problems, for example for improved upper bounds on Ramsey numbers through flag algebras, as recently done by Lidicky and Pfender [25]. The improvements however are largely not applicable when there are no previously ignored symmetries in the problem statement, as is for example the case with the famous $(3, 4)$-Turán conjecture. They may also not be helpful for applications beyond graphs [1, 2, 37, 34], where there can be more drastic jumps on the numbers of constraints as N is increased.

Besides these computational improvements, it is notable that our generalization of the stability argument from [7] no longer requires explicit knowledge of the Ramsey colorings underlying the extremal construction. While in our case the colorings were both known and crucial in order to derive an exact rather than a floating point-based flag algebra certificate, our main stability result draws a connection between the Ramsey number $R_{c-1}(3)$ and the Ramsey multiplicity problem $m_c(3)$, in theory opens up an avenue to establish a sort of equivalence of the two problems without first explicitly solving both or even either problem:

1) We could derive a flag algebra certificate for a particular $c > 4$ establishing $m_c(3)$ and meeting the necessary requirements without explicit knowledge of the $R_{c-1}(3)$-Ramsey colorings. Note that this would imply the exact value of $R_{c-1}(3) = m_c(3)^{-1/2} - 1$.

2) We could show that the Ramsey multiplicity problem satisfies the necessary requirements for arbitrary $c \geq 3$, in particular that $\hat{K}_{3,1}$ and $\hat{K}_{3,3}$ have zero density in an extremal construction, through a purely theoretical argument not relying on the semidefinite programming method and without explicitly determining $m_c(3)$. This would imply that $m_c(3) = (R_{c-1}(3) - 1)^{-2}$ without giving us explicit knowledge of either value.

It should be noted that Fox and Wigerson [13] somewhat recently characterized an infinite family of 2-colorings for which an upper bound equivalent to the one given by Equation (2) is tight, i.e., Turán graphs determine the extremal constructions for the respective Ramsey multiplicity problem. They also obtained results for the case of $c = 3$ colors that are conditioned the conjecturec bound $R(t, \lceil t/2 \rceil) \leq 2^{-31} R(t,t)$. At the risk of extrapolating from a sample size of two, this fact motivates us to go so far as to conjecture the following to be true.

Conjecture 6. *For any $c \geq 3$, we have $m_c(3) = (R_{c-1}(3) - 1)^{-2}$ and the only extremal constructions are derived from $R_{c-1}(3)$-Ramsey colorings.*

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