Totally Greedy Sequences Generated by a Class of Second-Order Linear Recurrences With Constant Coefficients

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Abstract

In the change-making problem the goal is to represent a certain amount of money with the least possible number of coins, chosen from a given set of denominations. The greedy algorithm picks the coin of largest possible denomination first. This strategy does not always produce the least number of coins, except when the set of denominations is endowed with certain properties, in which case it is called a greedy set. If the set of denominations is an infinite sequence, we call it totally greedy if every prefix subset is greedy. This paper investigates some totally greedy sequences generated by second-order linear recurrences with constant coefficients. In particular it investigates sufficient conditions for the sequence to be totally greedy.

1 Greedy sets and totally greedy sequences

In the *change-making problem* we are given a set of coin denominations $S = \{s_1 = 1, s_2, \ldots, s_t\}$, with $s_1 < \ldots < s_t$, and a target amount $k \in \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of nonnegative integers). The goal is to represent k using as few coins as possible from the given denominations. Mathematically, we are looking for a *payment vector* $(a_1, \ldots, a_t)$, such that: 1. $a_i \in \mathbb{N}_0$ for all $i = 1, \ldots, t$, 2. $\sum_{i=1}^{t} a_i s_i = k$, and 3. $\sum_{i=1}^{t} a_i$ is minimal.

This problem has been extensively studied in recent years (see for instance [1, 2, 3, 11]), and it is related to other problems involving integers, such as the *Frobenius problem* and the *postage stamp problem* [10]. It is also a special case of the well known *knapsack problem* [5]. Regarding its computational complexity, finding the optimal payment vector for a given k is NP-hard if the coins are large and represented in binary (or decimal) [4].

A simple approach for dealing with the problem is the *greedy algorithm*, which proceeds by first choosing the coin of the largest possible denomination, then the second largest, and so on. This idea is formalized in Algorithm 1:

Algorithm 1: GREEDY PAYMENT METHOD

Input : The set of denominations $S = 1, s_2, \ldots, s_t$, with $1 < s_2 < \ldots < s_t$, and a quantity $k \geq 0$.

Output: Payment vector $(a_1, a_2, \ldots, a_t)$.

for $i := t$ **downto** 1 **do**
  $a_i := k$ div s_i;
  $k := k$ mod s_i;
end

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Definition 1. *For a given set of denominations $S = 1, s_2, \dots, s_t$, the* greedy payment vector *is the payment vector $(a_1, a_2, \dots, a_t)$ produced by Algorithm 1, and* $\textsc{GreedyCost}_S(k) = \sum_{i=1}^{t} a_i$.

It is easy to verify that the greedy payment vector is not necessarily optimal (i.e. $\textsc{GreedyCost}_S(k)$ is not always minimal among all possible payment vectors) but there do exist some specific sets of denominations S for which the greedy payment vector is always optimal:

Definition 2. *If a set S of denominations* always *produces an optimal payment vector for* any *given amount k, then S is called* orderly*,* canonical*, or* greedy*.*

Greedy sets can be used, for instance, to construct circulant networks with efficient routing algorithms [8]. There is a polynomial-time algorithm that determines whether a given set of denominations is greedy [7, 10], as well as necessary and/or sufficient conditions for special families of denomination sets [1, 2, 3, 11]. The current paper continues along that path.

Note that a set S consisting of one or two denominations is always greedy. For sets of cardinal 3 we have the following [1]:

Proposition 3. *The set $S = \{1, a, b\}$ (with $a < b$) is greedy if, and only if, $b - a$ belongs to the set*

$$\mathfrak{D}(a) = \{a-1, a\} \cup \{2a-2, 2a-1, 2a\} \cup \dots \{ma - m, \dots ma\} \cup \dots = \bigcup_{m=1}^{\infty} \bigcup_{s=0}^{m} \{ma - s\}$$

□

The *one-point theorem* provides a powerful necessary and sufficient condition (Theorem 2.1 [1]):

Theorem 4. *Suppose that $S = \{1, s_2, \dots, s_t\}$ is a greedy set of denominations, and $s_{t+1} > s_t$. Now let $m = \left\lceil \frac{s_{t+1}}{s_t} \right\rceil$. Then $\hat{S} = \{1, s_2, \dots, s_t, s_{t+1}\}$ is greedy if, and only if,* $\textsc{GreedyCost}_S(ms_t - s_{t+1}) < m$.
□

Notice that

$$(m-1)s_t + 1 \le s_{t+1} \le ms_t,$$

by the definition of m. A straightforward consequence of the one-point theorem is the following

Corollary 5. *[Lemma 7.4 of [1]] Suppose that $S = \{1, s_2, \dots, s_t\}$ is a greedy set, and $s_{t+1} = us_t$, for some $u \in \mathbb{N}$. Then $\hat{S} = \{1, s_2, \dots, s_t, s_{t+1}\}$ is also greedy.* □

Definition 6. *A set $S = \{1, s_2, \dots, s_t\}$ is* totally greedy[1] *if every prefix subset $\{1, s_2, \dots, s_k\}$, with $k \le t$ is greedy.*

Obviously, a totally greedy set is also greedy, but the converse is not true in general. Take, for instance, the greedy set $\{1, 2, 5, 6, 10\}$, whose prefix subset $\{1, 2, 5, 6\}$ is not greedy.

Definition 6 can be extended to infinite sequences in a straightforward way:

Definition 7. *Let $S = \{s_n\}_{n=1}^{\infty}$ be an integer sequence, with $s_1 = 1$ and $s_i < s_{i+1}$ for all $i \in \mathbb{N}$. We say that S is* totally greedy *(or simply, greedy) if every prefix subset $\{1, s_2, \dots, s_k\}$ is greedy.*

Totally greedy sequences are briefly mentioned in [3], where some sufficient conditions are also given, that allow to construct greedy sequences from recurrences, although the conditions are a bit cumbersome (see Corollary 2.12 of [3]). Here we provide a simpler set of sufficient conditions that produce greedy sequences from second-order homogeneous recurrences.

[1] Also called normal, or totally orderly.

2 Sequences of the form $G_{n+2} = pG_{n+1} + qG_n$

We will consider sequences $\{G_n\}_{n=1}^{\infty}$ generated by the recurrence

$$G_n = \begin{cases} 1 \text{ if } n = 1, \\ a \text{ if } n = 2, \\ pG_{n-1} + qG_{n-2}, \text{ if } n > 2, \end{cases} \tag{1}$$

where a, p, q are positive integers, with $a > 1$, and some additional restrictions that we will see later on.

The (shifted) Fibonacci sequence $\{F_n\}_{n=1}^{\infty}$, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, is a special case of Equation 1, taking $a = p = q = 1$. Similarly, the Lucas numbers (Sequence A000032 of [6]) and the Pell numbers (Sequence A000129 of [6]) are also special cases of Equation 1.

The characteristic polynomial associated with Equation 1 is $x^2 - px - q$, and its roots are

$$\lambda = \frac{1}{2}\left(p + \sqrt{p^2 + 4q}\right), \qquad \mu = \frac{1}{2}\left(p - \sqrt{p^2 + 4q}\right), \tag{2}$$

with $\mu + \lambda = p$ and $\mu\lambda = -q$. Since the roots λ and μ are real and distinct, the general term of $\{G_n\}_{n=1}^{\infty}$ is

$$G_{n+1} = c_1\lambda^n + c_2\mu^n, \tag{3}$$

where $c_1 = \dfrac{a-\mu}{\lambda-\mu}$ and $c_2 = \dfrac{\lambda - a}{\lambda - \mu}$.

It is quite easy to see that $\{G_n\}_{n=1}^{\infty}$ is monotonically increasing, $|\lambda| > |\mu|$, and $\lambda > 1$. Moreover, it can be easily shown that $\lambda > p$ and $\mu < 0$. Now, if $q \leq p$ we can bound the roots λ and μ with more precision.

Lemma 8. *If $\{G_n\}_{n=1}^{\infty}$ is a sequence defined by Equation 1, with $q \leq p$, and λ and μ are the roots of the characteristic polynomial, as defined in Equation 2, then*

$$-1 < \mu < 0 \quad \text{and} \quad p < \lambda < p + 1.$$

Proof: Straightforward.

□

Note that as a consequence of the above results, c_1 is always positive, while c_2 can be positive or negative, depending on a. From now on, sequences that obey Equation 1, with $q \leq p$, will also be called *type-1-sequences*, and they will be the main focus of this section.

Now, in order to apply Theorem 4 we have to investigate the ratio

$$\frac{G_{n+1}}{G_n} = \frac{c_1\lambda^n + c_2\mu^n}{c_1\lambda^{n-1} + c_2\mu^{n-1}}, \tag{4}$$

where $\{G_n\}_{n=1}^{\infty}$ is a type-1-sequence.

Dividing the numerator and the denominator by λ^{n-1} we get

$$\frac{G_{n+1}}{G_n} = \frac{c_1\lambda + c_2\mu\left(\frac{\mu}{\lambda}\right)^{n-1}}{c_1 + c_2\left(\frac{\mu}{\lambda}\right)^{n-1}}. \tag{5}$$

Since $\left|\dfrac{\mu}{\lambda}\right| < 1$, $\left(\dfrac{\mu}{\lambda}\right)^{n-1} \longrightarrow 0$, and

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \lambda \in (p, p+1). \tag{6}$$

It will also be useful (and instructive) to investigate how the different subsequences of $\left\{\dfrac{G_{n+1}}{G_n}\right\}$ approach the limit value of λ.

Lemma 9. *Let $\{G_n\}_{n=1}^{\infty}$ be a type-1-sequence. Then*

1. *If $a < \lambda$ (respectively $a > \lambda$) the subsequence $\left\{\frac{G_{2k+2}}{G_{2k+1}}\right\}_{k=0}^{\infty}$ is monotonically increasing (respectively decreasing).*

2. *If $a < \lambda$ (respectively $a > \lambda$) the subsequence $\left\{\frac{G_{2k+1}}{G_{2k}}\right\}_{k=1}^{\infty}$ is monotonically decreasing (respectively increasing).*

Proof: One way of proving the monotonicity of the subsequence $\left\{\frac{G_{2k+2}}{G_{2k+1}}\right\}$ is by investigating the difference

$$\frac{G_{2k+2}}{G_{2k+1}} - \frac{G_{2k+4}}{G_{2k+3}} = \frac{G_{2k+2}G_{2k+3} - G_{2k+1}G_{2k+4}}{G_{2k+1}G_{2k+3}} \tag{7}$$

in the first case, and the difference

$$\frac{G_{2k+1}}{G_{2k}} - \frac{G_{2k+3}}{G_{2k+2}} = \frac{G_{2k+1}G_{2k+2} - G_{2k}G_{2k+3}}{G_{2k}G_{2k+2}} \tag{8}$$

in the second case, i.e. in the subsequence $\left\{\frac{G_{2k+1}}{G_{2k}}\right\}$. Since both denominators are positive, we will investigate the sign of the numerators

$$G_{2k+2}G_{2k+3} - G_{2k+1}G_{2k+4} = c_1c_2\lambda^{2k}\mu^{2k}\left(\lambda\mu^2 + \lambda^2\mu - \mu^3 - \lambda^3\right) \tag{9}$$

and

$$G_{2k+1}G_{2k+2} - G_{2k}G_{2k+3} = c_1c_2\lambda^{2k-1}\mu^{2k-1}\left(\lambda\mu^2 + \lambda^2\mu - \mu^3 - \lambda^3\right), \tag{10}$$

respectively.

In the first case, the sign of the expression (9) depends solely on c_2, since c_1, λ^{2k}, and μ^{2k} are all positive, while $\left(\lambda\mu^2 + \lambda^2\mu - \mu^3 - \lambda^3\right) = -p\left(p^2 + 4q\right)$ is negative. If $a < \lambda$, then $c_2 > 0$, and (9) is negative, which means that $\left\{\frac{G_{2k+2}}{G_{2k+1}}\right\}$ is increasing. On the other hand, if $a > \lambda$, then $c_2 < 0$, and (9) is positive, which means that $\left\{\frac{G_{2k+2}}{G_{2k+1}}\right\}$ is decreasing.

In the second case, the sign of the expression (10) again depends solely on c_2, since c_1 and λ^{2k-1} are positive, while μ^{2k-1} and $\left(\lambda\mu^2 + \lambda^2\mu - \mu^3 - \lambda^3\right)$ are negative. The rest is similar.

□

Corollary 10. *Let $\{G_n\}_{n=1}^{\infty}$ be a type-1-sequence. Then there exists an integer $2 \le K_0 \le 3$ such that for all $n \ge K_0$ we have*

$$\frac{G_{n+1}}{G_n} \in (p, p+1) \tag{11}$$

Proof: We just have to check that $2 \le K_0 \le 3$. For all $n \ge 3$ we have

$$\frac{G_{n+1}}{G_n} = \frac{pG_n + qG_{n-1}}{G_n} = p + \frac{qG_{n-1}}{pG_{n-1} + qG_{n-2}} \in (p, p+1),$$

since $q \le p$ and $qG_{n-2} > 0$. Hence, $K_0 \le 3$.

Now, if additionally $a > q$, then $\frac{G_3}{G_2} = p + \frac{q}{a} \in (p, p+1)$, hence $K_0 = 2$.

□

Let's denote the prefix set $\{1, G_2, \ldots, G_k\}$ of $\{G_n\}_{n=1}^{\infty}$ by $G^{(k)}$. We know that $G^{(2)} = \{1, a\}$ is always greedy, and we will now investigate when $G^{(3)}$ is greedy:

Lemma 11. *Let $\{G_n\}_{n=1}^{\infty}$ be a type-1-sequence, then $G^{(3)} = \{1, a, pa + q\}$ is (totally) greedy if, and only if, $2 \leq a \leq p + q$.*

Proof: By Proposition 3, the set $\{1, a, pa + q\}$ is greedy if and only if $pa + q - a$ belongs to the set

$$\mathfrak{D}(a) = \{a - 1, a\} \cup \{2a - 2, 2a - 1, 2a\} \cup \ldots \{ma - m, \ldots ma\} \cup \ldots.$$

If $a > p + q$ then $pa + q - a \notin \mathfrak{D}(a)$, so $G^{(3)}$ is not greedy. Hence $2 \leq a \leq p + q$. Let us now check that this condition is sufficient.

We may split the condition $2 \leq a \leq p + q$ into two cases:

1. $a < q$, and
2. $q \leq a \leq p + q$.

In the second case it is easy to see that $pa + q - a \in \mathfrak{D}(a)$, hence $G^{(3)}$ is greedy. In the first case let $m' = \left\lceil \frac{q}{a} \right\rceil > 1$.

$$\begin{aligned} pa + q - a &= pa + q - a + (m' - 1)a - (m' - 1)a \\ &= (p + m' - 1)a - (m'a - q). \end{aligned}$$

Thus, $pa + q - a \in \mathfrak{D}(a)$ if, and only if, $0 \leq m'a - q \leq p + m' - 1$. We already know that $m'a - q \geq 0$ by the definition of m'. As for the other inequality, we have

$$m'a - q < 2q - q = q \leq p < p + m' - 1.$$

□

Now we are in the position to prove our main result:

Theorem 12. *Let $\{G_n\}_{n=1}^{\infty}$ be a type-1-sequence with $2 \leq a \leq p + q$. Then $\{G_n\}_{n=1}^{\infty}$ is totally greedy.*

Proof: The theorem is proved by induction. Lemma 11 guarantees that $G^{(3)}$ is greedy; that would be the base case. Now, let's suppose that $G^{(k)}$ is totally greedy for some arbitrary $k \geq 3$. We will prove that $G^{(k+1)}$ is also greedy (and hence totally greedy).

By Lemma 8 and Corollary 10 we know that $p < \frac{G_{k+1}}{G_k} < p + 1$, so $m = \left\lceil \frac{G_{k+1}}{G_k} \right\rceil = p + 1$. Now,

$$\begin{aligned} (p + 1)G_k - G_{k+1} &= (p + 1)G_k - (pG_k + qG_{k-1}) \\ &= G_k - qG_{k-1} = (pG_{k-1} + qG_{k-2}) - qG_{k-1} \\ &= (p - q)G_{k-1} + qG_{k-2}. \end{aligned}$$

To conclude the proof, note that $\textsc{GreedyCost}_{G^{(k)}}((p - q)G_{k-1} + qG_{k-2}) = p - q + q = p < p + 1 = m$.

□

We can now apply Theorem 12 to some specific sequences, such as the (shifted) Fibonacci numbers $\{F_n\}_{n=1}^{\infty} = \{1, 2, 3, 5, 8, 13, \ldots\}$, and the (shifted) Pell numbers $\{P_n\}_{n=1}^{\infty} = \{1, 2, 5, 12, 29, 70 \ldots\}$.

A full version of this paper, including these and other results, can be found in [9].

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