On a conjecture concerning the roots of Ehrhart polynomials of symmetric edge polytopes from complete multipartite graphs*

Max Kölbl[†1]

[1]Dept. of Pure and Applied Mathematics, Osaka University, 565-0871 Suita

Abstract

In [7], Higashitani, Kummer, and Michałek pose a conjecture about the symmetric edge polytopes of complete multipartite graphs and confirm it for a number of families in the bipartite case. We confirm that conjecture for a number of new classes following the authors' methods and we present a more general result which suggests that the methods in their current form might not be enough to prove the conjecture in full generality.

1 Introduction

This paper is an extended abstract of our recent work [8] for the Discrete Mathematics Day 2024. It contains the main results from Sections 3 and 4.

A *lattice polytope* is a convex polytope $P \subset \mathbb{R}^n$ which can be written as the convex hull of finitely many elements of $\mathbb{Z}^n$. Lattice polytopes arise naturally from attempts to endow combinatorial objects with a geometric structure. A family of lattice polytopes that has garnered some attention in recent years is that of *symmetric edge polytopes*, which are a type of graph polytopes. For graphs, we will henceforth use the notation $G = (V, E)$ where V denotes the set of *vertices* and E denotes the set of *edges* of G. Given a graph $G = (V, E)$, we thus define its symmetric edge polytope as follows

$$P_G = \operatorname{conv}\{\pm(e_v - e_w) \colon \{v, w\} \in E\} \subset \mathbb{R}^{|V|}.$$

Here, the vectors e_v are elements that form a lattice basis of $\mathbb{Z}^{|V|}$. For more context on symmetric edge polytopes, see e.g. [6, 9].

Next, we define the *lattice-point enumerator* of a set $S \subset \mathbb{R}^n$ as the function $E_S \colon \mathbb{N} \to \mathbb{N}$ via $E_S(k) = |kS \cap \mathbb{Z}^n|$. If S is a lattice polytope, E_S is a polynomial which we call the *Ehrhart polynomial* of S. The generating function of an Ehrhart polynomial is called *Ehrhart series* and can be written as

$$\operatorname{ehr}_P(t) = \sum_{k \geq 0} E_P(k) t^k = \frac{h^*(t)}{(1-t)^{d+1}},$$

where $h^*(t)$ is a polynomial with non-negative integer coefficients of degree d or less. We call this polynomial the *h^*-polynomial* of P. The Ehrhart polynomial and the h^*-polynomial hold valuable information about the underlying polytope, such as its (normalised) volume and the volume of its boundary. A specifically remarkable piece of information encoded by the h^*-polynomial is that of reflexivity: A lattice polytope is called *reflexive* if its polar dual is also a lattice polytope. By a result by Hibi [5], P is reflexive if and only if its h^*-polynomial is *palindromic*, i.e., $h^*_P(t) = \sum_{i=0}^{d} h^*_i(t)$ satisfies $h^*_i = h^*_{d-i}$ for all $0 \leq i \leq d$, and its degree is equal to $\dim P$.

*The full version of this work can be found in [8] and will be published elsewhere.

†Email: max.koelbl@ist.osaka-u.ac.jp

With some basic knowledge of generating functions (see e.g. [12]), one can check that knowing the Ehrhart polynomial of a lattice polytope amounts to knowing its h^*-polynomial. However, the converse is also true. Given the h^*-polynomial $h^*_P(t) = \sum_{i=0}^{d} h^*_i t^i$ of some lattice polytope P, the Ehrhart polynomial can be written as

$$E_P(x) = \sum_{i=0}^{d} h^*_i \binom{d+x-i}{d}.$$

For more context on Ehrhart theory, see e.g. [1]. One aspect of research in Ehrhart theory is the study of the *roots* of Ehrhart polynomials when their domain and range are extended from $\mathbb{N}$ to $\mathbb{C}$. For example in the case of reflexive polytopes, their Ehrhart polynomial roots exhibit symmetry not only across the real axis (i.e. if z is a root then so is its complex conjugate) but also, due to Ehrhart-Macdonald reciprocity and palindromicity of the h^*-polynomial, across the *canonical line*, i.e. set

$$\mathrm{CL} = \left\{ z \in \mathbb{C} \colon \Re(z) = -\frac{1}{2} \right\}$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. This is to say that if z is a root then so is $-1-z$. Thus, it is natural to ask, what kind of polytopes have all of their Ehrhart polynomial roots *on* CL. First steps in this direction were made in [2] and [10], albeit in different contexts. In [9], the study of CL*-polytopes*, i.e., polytopes with all their Ehrhart polynomial roots on CL, has been initiated as a field of study in its own right. For low dimensions, a full classification was found in [4]. Some classes of examples include cross-polytopes, standard reflexive simplices, and root polytopes of type A.

For the rest of the paper, let $K_{a_1,\dots,a_k}$ denote the complete multipartite graph with k multipartite classes of of sizes a_1 through a_k. The Ehrhart polynomial of $P_{K_{a_1,\dots,a_k}}$ shall be denoted by $E_{a_1,\dots,a_k}$. In [7], the authors studied the roots of $E_{2,n}$ and $E_{3,n}$ and were able to prove that $P_{K_{2,n}}$ and $P_{K_{3,n}}$ are CL-polytopes. This extends the case of cross-polytopes, which are unimodularly equivalent to the symmetric edge polytopes of $K_{1,n}$. They accomplished that by using the technique of *interlacing polynomials.* Let f, g be polynomials of degree $d+1$ and d with roots $\left\{-\frac{1}{2} + i\,a_1, -\frac{1}{2} + i\,a_2, \dots, -\frac{1}{2} + i\,a_{d+1}\right\}$ and $\left\{-\frac{1}{2} + i\,b_1, -\frac{1}{2} + i\,b_2, \dots, -\frac{1}{2} + i\,b_d\right\}$ respectively for $a_j, b_j \in \mathbb{R}$. Then we say that g CL*-interlaces* f if

$$a_1 \leq b_1 \leq a_2 \leq b_2 \leq \cdots \leq b_d \leq a_{d+1}.$$

For more on the theory of interlacing polynomials, see [3]. The authors gave the following conjecture.

Conjecture 1 (Conjecture 4.10 in [7])**.** *(i) For any complete multipartite graph $K_{a_1,\dots,a_k}$ the Ehrhart polynomial $E_{a_1,\dots,a_k}$ has its roots on* CL*.*

(ii) Suppose $a_1 \leq \cdots \leq a_k$. The two Ehrhart polynomials $E_{a_1,\dots,a_k}$ and $E_{a_1,a_2,\dots,a_k-1}$ interlace on CL*.*

In Section 2, we will prove CL-ness of $E_{1,1,n}, E_{1,2,n}$, and $E_{1,1,1,n}$, as well as some conditional results (Theorem 9), using the techniques from [7]. In Section 3, we will investigate the connection between the γ*-vector* of the h^*-polynomial of an Ehrhart polynomial and the existence of recursive relations that generalise those in [7]. However, we also provide evidence for why their methods might not be enough to prove Conjecture 1 any further.

2 New recursive relations

In this section, we gather new evidence for Conjecture 1. First, we state the relevant h^*-polynomials.

Proposition 2 (Theorem 4.1 in [6])**.** *For all $a, b \geq 0$ let $h^*_{a,b}(t)$ denote the h^*-polynomial of the symmetric edge polytope of $K_{a+1,b+1}$. Then*

$$h^*_{a,b}(t) = \sum_{i=0}^{\min\{a,b\}} \binom{2i}{i}\binom{a}{i}\binom{b}{i} t^i (1+t)^{a+b+1-2i}.$$

Proposition 3. *The h^*-polynomials of the symmetric edge polytopes of the graphs $K_{1,m,n}$, $K_{1,1,1,n}$, and $K_{2,2,n}$, are given as follows.*

(a) $h^*_{1,m,n}(t) = \sum_{i=0}^{\min(m,n)} \binom{2i}{i}\binom{m}{i}\binom{n}{i} t^i (1+t)^{m+n-2i}$

(b) $h^*_{1,1,1,n}(t) = 3(n-1)n(1+t)^{n-2}t^2 + 2(2n+1)(1+t)^n t + (1+t)^{n+2}$

(c) $h^*_{2,2,n}(t) = 20\binom{n}{3}(1+t)^{n-3}t^3 + 2\binom{3n}{2}(1+t)^{n-1}t^2 + 2\binom{3n+1}{1}(1+t)^{n+1}t + (1+t)^{n+3}$

Since the proof is very technical, we will proceed directly to introducing a proposition which supplies a useful tool for checking CL-interlacing.

Proposition 4 (Lemmas 2.3, 2.4, 2.5 in [7])**.** *Let $f, g, h_1, \dots, h_n$ be Ehrhart polynomials of reflexive polytopes such that* $\deg f = \deg g + 1 = \deg h_i + 2$ *for all $1 \le i \le n$. Assume the identity*

$$f(x) = (2x+1)\,\alpha\, g(x) + \sum_{i=1}^{n} \alpha_i\, h_i(x)$$

where $\alpha, \alpha_i > 0$ for all i. Then $\sum_{i=1}^{n} \alpha_i\, h_i$ CL-*interlaces g if for every i, h_i* CL-*interlaces g. Also, the following are equivalent.*

(a) $\sum_{i=1}^{n} \alpha_i\, h_i$ CL-*interlaces g,*

(b) g CL-*interlaces f.*

If this is the case, $(2x+1)\sum_{i=1}^{n} \alpha_i\, h_i$ CL-*interlaces f.*

An important class of reflexive polytopes is the class of *cross-polytopes* which are defined as the convex hull of the vectors $\pm e_1, \pm e_2, \dots, \pm e_n \in \mathbb{R}^n$. As mentioned in the introduction, they are unimodularly equivalent to $P_{K_{1,n}}$. The Ehrhart polynomial of the n-dimensional cross-polytope (the n-th *cross-polynomial*) is given by

$$\mathcal{C}_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+x-k}{n}.$$

Cross-polynomials are the first class of examples to showcase the usefulness of Proposition 4.

Proposition 5 (Example 3.3 in [7])**.** *For any $n \ge 2$, cross-polynomials satisfy the recursive relation*

$$\mathcal{C}_n(x) = \frac{1}{n}\,(2x+1)\,\mathcal{C}_{n-1}(x) + \frac{n-1}{n}\,\mathcal{C}_{n-2}(x).$$

Other classes of examples were found by Higashitani, Kummer, and Michałek in [7]. The authors found three recursive relations among Ehrhart polynomials $E_{1,n}, E_{2,n}, E_{3,n}$.

Proposition 6 (Proposition 4.5 in [7])**.** *The following relations hold:*

$$\begin{aligned}
E_{2,n}(x) &= \frac{1}{2}\,(2x+1)\,E_{1,n}(x) + \frac{1}{2}\,E_{1,n-1}(x),\\
E_{2,n}(x) &= \frac{1}{n}\,(2x+1)\,E_{2,n-1}(x) + \frac{1}{2n}\,\left(n\,E_{1,n-1}(x) + (n-2)\,(2x+1)\,E_{1,n-2}(x)\right),\\
E_{3,n+1}(x) &= \frac{(2x+1)(3n^2+13n+16)}{8(n^2+5n+6)}\,E_{2,n+1}(x)\\
&\quad + \frac{n^3+13n^2+18n}{8(n-1)(n^2+5n+6)}\,E_{2,n}(x) + \frac{4n^3+9n^2-13n-32}{8(n-1)(n^2+5n+6)}\,E_{1,n+1}(x).
\end{aligned}$$

Using this, the authors were able to prove the following result.

Proposition 7 (Lemmas 4.6-4.8, Theorem 4.9 in [7]). *The following statements hold for every positive integer n.*

(a) $E_{1,n}$ CL-*interlaces* $E_{1,n+1}$.

(b) $E_{1,n}$ *and* $(2x+1)E_{1,n-1}$ CL-*interlace* $E_{2,n}$.

(c) $E_{2,n}$ CL-*interlaces* $E_{2,n+1}$.

(d) $E_{2,n}$ CL-*interlaces* $E_{3,n}$.

In particular, for every positive integer n, the Ehrhart polynomial of $K_{m,n}$ is a CL-*polynomial if $m \leq 2$.*

To extend this result, we start by finding new recursive relations.

Proposition 8. *For every $n \geq 2$ there exist non-negative rational numbers $\alpha_1, \dots, \alpha_{35}$ such that the following statements hold.*

$$\begin{aligned}
E_{1,1,n}(x) &= \alpha_1\,(2x+1)\,E_{1,n}(x) + \alpha_2\,E_{1,n-1}(x),\\
E_{1,1,n+1}(x) &= \alpha_3\,(2x+1)\,E_{1,1,n}(x) + \alpha_4\,E_{1,1,n-1}(x) + \alpha_5\,E_{1,n}(x),\\
E_{1,2,n}(x) &= \alpha_6\,(2x+1)\,E_{1,1,n}(x) + \alpha_7\,E_{1,1,n-1}(x) + \alpha_8\,E_{1,n}(x),\\
E_{1,2,n+1}(x) &= \alpha_9\,(2x+1)\,E_{1,2,n}(x) + \alpha_{10}\,E_{1,2,n-1}(x) + \alpha_{11}\,E_{1,1,n}(x) + \alpha_{12}\,E_{1,n+1}(x)\\
E_{1,1,1,n}(x) &= \alpha_{13}\,(2x+1)\,E_{1,1,n}(x) + \alpha_{14}\,E_{1,1,n-1}(x) + \alpha_{15}\,E_{1,n}(x)\\
E_{4,n}(x) &= \alpha_{16}\,(2x+1)\,E_{3,n}(x) + \alpha_{17}\,E_{3,n-1}(x) + \alpha_{18}\,E_{2,n}(x) + \alpha_{19}\,E_{1,n+1}(x),\\
E_{3,n+1}(x) &= \alpha_{20}\,(2x+1)\,E_{3,n}(x) + \alpha_{21}\,E_{3,n-1}(x) + \alpha_{22}\,E_{2,n}(x) + \alpha_{23}\,E_{1,n+1}(x),\\
E_{2,2,n}(x) &= \alpha_{24}\,(2x+1)\,E_{1,2,n}(x) + \alpha_{25}\,E_{1,2,n-1}(x) + \alpha_{26}\,E_{1,1,n}(x) + \alpha_{27}\,E_{1,n+1}(x),\\
E_{1,3,n}(x) &= \alpha_{28}\,(2x+1)\,E_{1,2,n}(x) + \alpha_{29}\,E_{1,2,n-1}(x) + \alpha_{30}\,E_{1,1,n}(x) + \alpha_{31}\,E_{1,n+1}(x),\\
E_{1,1,1,n+1}(x) &= \alpha_{32}\,(2x+1)\,E_{1,1,1,n}(x) + \alpha_{33}\,E_{1,1,1,n-1}(x) + \alpha_{34}\,E_{1,1,n}(x) + \alpha_{35}\,E_{1,n+1}(x).
\end{aligned}$$

These relations can be obtained algorithmically. We explain the method using the first identity as an example. The identity holds if and only if it holds after replacing $E_{1,1,n}(x)$, $(2x+1)E_{1,n}(x)$, and $E_{1,n-1}(x)$ by their respective generating functions. All three of these can be obtained from h^*-polynomials given in Propositions 2 and 3. After dividing by the left-hand side, the right hand side becomes a rational function whose numerator polynomial has coefficients which are either constant or linear in α_1 and α_2. The left-hand side becomes 1. Thus, on the right-hand side, we can compare the coefficients of the numerator polynomial with those of the denominator polynomial and solve for α_1 and α_2. Note however, that in general there need not be a solution. A SAGEMATH [11] implementation of this algorithm is available on

`https://github.com/maxkoelbl/seps_multipartite_graphs/`.

We can state the main result of this section.

Theorem 9. *The following statements hold for every positive integer n.*

(a) $E_{1,n}$ CL-*interlaces* $E_{1,1,n}$.

(b) $E_{1,1,n}$ CL-*interlaces* $E_{1,1,n+1}$.

(c) $E_{1,1,n}$ CL-*interlaces* $E_{1,2,n}$.

(d) $E_{1,1,n}$ CL-*interlaces* $E_{1,1,1,n}$.

(e) $E_{3,n}$ CL-*interlaces* $E_{4,n}$ *if* $E_{1,n+1}$ CL-*interlaces* $E_{3,n}$.

(f) $E_{1,2,n}$ CL-*interlaces* $E_{1,3,n}$ *if* $E_{1,n+1}$ CL-*interlaces* $E_{1,2,n}$.

(g) $E_{1,2,n}$ CL-*interlaces* $E_{2,2,n}$ *if* $E_{1,n+1}$ CL-*interlaces* $E_{1,2,n}$.

In particular, for every positive integer n, $E_{x,y,z,n}$ is a CL-*polynomial for $x+y+z \leq 3$ and $x, y, z \geq 0$.*

3 Recursive relations and the γ-vector

Looking at the recursive relations in Propositions 6 and 8, we may notice that as the parameters $a_1, \ldots, a_{k-1}$ of the multipartite graphs increase, then so does the complexity of the identities involving them. The results of this section show that this is not a coincidence. The key object here is the *γ-vector* of the h^*-polynomial of an Ehrhart polynomial.

Definition 10. *Let h be a palindromic polynomial of degree d. We define the γ-*vector *as the polynomial $\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i$ such that $h(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i (1+t)^{d-2i} t^i$. We call the degree of the γ-vector the γ-*degree *of h.*

Proposition 11. *Let p be a polynomial of degree d and let h be a polynomial defined by*

$$h(t) = (1-t)^{d+1} \sum_{k \geq 0} p(k)\, t^k.$$

If h is a palindromic polynomial with γ-vector γ, we get

$$p(x) = \sum_{i=0}^{\deg \gamma} (-1)^i c_i \, \mathcal{C}_{d-2i}(x).$$

where $c_i = \sum_{j=i}^{\deg \gamma} \frac{1}{4^j} \binom{j}{i} \gamma_j$.

In the setting of Proposition 11, we call the γ-degree of h the *cross-degree* of p. It is the key ingredient of this section's main theorem.

Theorem 12. *Let f be a degree $d+1$ polynomial with cross-degree $m+1$, let g be a degree d polynomial with cross-degree m, and let h_i be degree $d-1$ polynomials with cross-degree i for $1 \leq i \leq m$. Then there exist unique real numbers $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_m$ which satisfy*

$$f(x) = (2x+1)\, \alpha\, g(x) + \sum_{i=1}^{m} \alpha_i \, h_i(x).$$

For complete bipartite graphs, Proposition 2 shows that the γ-degree of the h^*-polynomial of $K_{m,n}$ is $\min\{m,n\} - 1$. Thus, we get the following an immediate corollary.

Corollary 13. *Let n be a positive integer. For $1 \leq m \leq n$ there exist unique $\alpha, \alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ and $\beta, \beta_0, \beta_1, \ldots, \beta_{m-1}$ in $\mathbb{R}$ such that the following equations are satisfied.*

$$E_{m+1,n+1}(x) = (2x+1)\, \alpha\, E_{m,n+1}(x) + \sum_{i=0}^{m-1} \alpha_i \, E_{m-i,n+i}(x)$$

$$E_{m,n+1}(x) = (2x+1)\, \beta\, E_{m,n}(x) + \sum_{i=0}^{m-1} \beta_i \, E_{m-i,n+i-1}(x)$$

This corollary alone is not enough to prove Conjecture 1 for all $K_{m,n}$ for two crucial reasons. Firstly, as m increases, the number of interlacings having to be satisfied increases too, and they are between polynomials whose cross-degrees puts them outside the scope of Theorem 12. This is noticeable in the last four statements of Theorem 9 where the interlacing of cross-degree 3 polynomials by cross-degree 2-polynomials depends on the interlacing of a cross-degree 2-polynomial by a cross-degree 0 polynomial. Secondly, there is no guarantee that the coefficients $\alpha, \alpha_1, \ldots, \alpha_m$ are non-negative. In fact, for $m \geq 4$, explicit computations reveal that $\alpha_2, \ldots, \alpha_{m-2}$ are always negative. In the case $m = 4$, we get $\alpha_2 = \frac{n-n^3}{8(5n^3+39n^2+100n+96)}$. To see the parameters for every $1 \leq m \leq 10$, we refer once again to the corresponding SAGEMATH code in the previously mentioned github repository.

We end by presenting a conjecture.

Conjecture 14. *Let $a_1 \leq a_2 \leq \cdots \leq a_k \leq n$ be positive integers and let m denote the cross-degree of the Ehrhart polynomial of the symmetric edge polytope of $K_{a_1,a_2,\ldots,a_k}$. Then the inequalities*

$$\left\lfloor \frac{\sum_{i=1}^{k} a_i}{2} \right\rfloor \leq m+1 \leq \sum_{i=1}^{k} a_i.$$

hold. Furthermore, the Ehrhart polynomial of the symmetric edge polytope of the graph $K_{1^k,n}$ interlaces that of $K_{1^{k+1},n}$, where 1^k represents a list k of ones. For $k+n \leq 10$, this has been computationally confirmed.

Acknowledgements

I would like to express my gratitude to my advisor Akihiro Higashitani and to Rodica Dinu for many useful discussions and invaluable advice.

References

[1] Matthias Beck and Sinai Robins. *Computing the continuous discretely.* Undergraduate Texts in Mathematics. Integer-point enumeration in polyhedra. Springer, New York, 2007, pp. xviii+226.

[2] Daniel Bump, Kwok-Kwong Choi, Pär Kurlberg, and Jeffrey Vaaler. "A local Riemann hypothesis. I". In: *Math. Z.* 233.1 (2000), pp. 1–19.

[3] S. Fisk. "Polynomials, roots, and interlacing". In: *arXiv Mathematics e-prints* (2006), math–0612833.

[4] Gábor Hegedüs, Akihiro Higashitani, and Alexander Kasprzyk. "Ehrhart polynomial roots of reflexive polytopes". In: *Electron. J. Combin.* 26.1 (2019), Paper No. 1.38, 27.

[5] Takayuki Hibi. "Dual polytopes of rational convex polytopes". In: *Combinatorica* 12.2 (1992), pp. 237–240.

[6] Akihiro Higashitani, Katharina Jochemko, and Mateusz Michałek. "Arithmetic aspects of symmetric edge polytopes". In: *Mathematika* 65.3 (2019), pp. 763–784.

[7] Akihiro Higashitani, Mario Kummer, and Mateusz Michałek. "Interlacing Ehrhart polynomials of reflexive polytopes". In: *Selecta Math. (N.S.)* 23.4 (2017), pp. 2977–2998.

[8] Max Kölbl. "On a Conjecture Concerning the Roots of Ehrhart Polynomials of Symmetric Edge Polytopes from Complete Multipartite Graphs". In: *arXiv preprint arXiv:2404.02136* (2024).

[9] Tetsushi Matsui, Akihiro Higashitani, Yuuki Nagazawa, Hidefumi Ohsugi, and Takayuki Hibi. "Roots of Ehrhart polynomials arising from graphs". In: *J. Algebraic Combin.* 34.4 (2011), pp. 721–749.

[10] Fernando Rodriguez-Villegas. "On the zeros of certain polynomials". In: *Proc. Amer. Math. Soc.* 130.8 (2002), pp. 2251–2254.

[11] The Sage Developers et al. *SageMath, version 10.2.* 2023. URL: `http://www.sagemath.org`.

[12] Herbert S. Wilf. *generatingfunctionology.* Third. A K Peters, Ltd., Wellesley, MA, 2006, pp. x+245.