On Ewald's and Nill's Conjectures about smooth polytopes*

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Abstract

A monotone polytope in $\mathbb{R}^n$ is a smooth reflexive polytope. These polytopes arise as the momentum polytopes of monotone symplectic toric manifolds. Ewald's well-known Conjecture from 1988 states that if P is a monotone n-polytope in $\mathbb{R}^n$ then the set $\mathbb{Z}^n \cap P \cap -P$ contains a unimodular basis of the lattice $\mathbb{Z}^n$. McDuff (2009) shows that a stronger property of a monotone polytope, which she calls *star Ewald condition*, is closely related to whether the central fiber of the corresponding monotone symplectic toric manifold is a stem. In 2009 Nill proposed a generalization of Ewald's Conjecture to smooth lattice polytopes. In this extended abstract, prepared for the Discrete Mathematics Days conference (University of Alcalá, July 3-5, 2024), we summarize the results concerning these conjectures that we have obtained in our recent article `arXiv:2310.10366`. We refer to this article for details and proofs.

1 Introduction

The goal of this extended abstract is to report on the results from our recent paper [4], which solves some broad cases of a well-known conjecture by G. Ewald from 1988 concerning monotone lattice polytopes [5], and its more recent generalization to smooth polytopes by B. Nill, from 2009 [13]. Our motivation comes partially from symplectic geometry, as we will explain, but for brevity we do not discuss our results in this direction. We refer to the original article [4] for more details, complete statements, and proofs.

Smooth polytopes in general, and smooth reflexive ones in particular, are very important in algebraic and symplectic geometry, providing a strong link between "discrete" problems in combinatorics/convex geometry and "continuous" problems concerning smooth (toric) manifolds. In fact, smooth reflexive n-dimensional polytopes are also known as *monotone n-dimensional polytopes*, as they are the images of $2n$-dimensional monotone symplectic toric manifolds under the momentum map $M \to \mathbb{R}^n$. We refer to Charton-Sabatini-Sepe [2], Godinho-Heymann-Sabatini [7] and McDuff [10], for recent works which discuss monotone polytopes from the perspective of symplectic geometry and to Batyrev [1], Cox-Little-Schenck [3, Theorem 8.3.4], Franco-Seong [6], Haase-Melnikov [8] and Nill [12] for their relation to Gorenstein Fano varieties in algebraic geometry.

Let us recall their precise definitions:

Definition 1 (Smooth polytope)**.** *An n-dimensional polytope P in $\mathbb{R}^n$ is* smooth *if it satisfies the following three properties:*

*The full version of this work can be found in [4] and will be published elsewhere.

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- *P is* simple*: there are precisely n edges meeting a each vertex;*
- *P is* rational*: it has rational edge directions (equivalently, the normal vector to the facets are rational);*
- *the primitive edge-direction vectors at each vertex of P form a basis of the lattice $\mathbb{Z}^n$.*

Equivalently, a smooth polytope P in $\mathbb{R}^n$ is a polytope whose normal fan is simplicial, rational, and unimodular.

Definition 2 (Reflexive polytope)**.** *A* reflexive polytope *in $\mathbb{R}^n$ is a lattice polytope with the origin in its interior and whose dual polytope is also a lattice polytope. Equivalently, a lattice polytope in $\mathbb{R}^n$ is reflexive if and only if every facet-defining inequality is of the form $u_F \cdot x \leqslant 1$, where u_F is the primitive exterior normal vector to the facet.*

Definition 3 (Monotone polytope)**.** *A polytope in $\mathbb{R}^n$ is* monotone *if it is smooth and reflexive.*

There are finitely many monotone polytopes in each dimension n modulo unimodular equivalence (that is, modulo $\mathrm{GL}(n, \mathbb{Z})$ or equivalently $\mathrm{AGL}(n, \mathbb{Z})$ transformations). Up to dimension 9 they are counted in [9, 14] and, as seen in Table 1, the number of monotone polytopes increases rapidly with the dimension. Figure 1 shows the five possibilities in dimension two.

dimension	1	2	3	4	5	6	7	8	9
monotone polytopes	1	5	18	124	866	7622	72256	749892	8229721

Table 1: Number of monotone polytopes in each dimension up to 9.

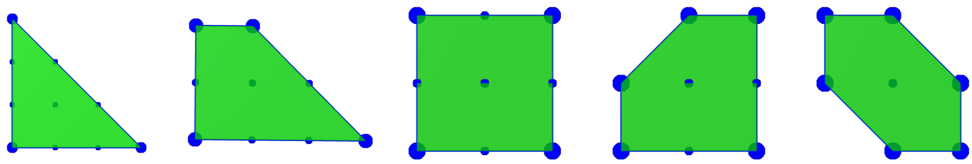

Figure 1: The five monotone polygons: monotone triangle, trapezoid, square, pentagon, and hexagon.

We are interested in understanding, both theoretically and computationally, the properties of the *Ewald set* of a monotone polytope. This set appears implicitly in the influential 1988 paper by Günter Ewald [5].

Definition 4 (Ewald set [4, Definition 1.1])**.** *The* Ewald set *of a polytope $P \subset \mathbb{R}^n$ is*

$$\mathcal{E}(P) := \mathbb{Z}^n \cap P \cap -P.$$

Its points are called Ewald points *of P.*

That is, $\mathcal{E}(P) \subset \mathbb{Z}^n$ consists of the *symmetric integral points of P*, meaning integral points $x \in \mathbb{Z}^n$ for which both $x \in P$ and $-x \in P$. Our main motivation is the following conjecture:[1]

Conjecture 5 (Ewald's Conjecture 1988 [5, Conjecture 2])**.** *Let $n \in \mathbb{N}$. If P is an n-dimensional monotone polytope in $\mathbb{R}^n$ then $\mathcal{E}(P)$ contains a unimodular basis of $\mathbb{Z}^n$.*

[1] The original formulation of Conjecture 5 refers to dual polytopes, stating that the dual of any monotone polytope P can be sent into $[-1, 1]^n$, via a unimodular transformation. As pointed out by Øbro [15] this is equivalent to our formulation, used already by McDuff [11, Section 3.1] and Payne [16, Remark 4.6]. (McDuff and Payne remove the origin from $\mathcal{E}(P)$ in their definition, but for technical reasons we do not).

The conjecture has been verified computationally for $n \leqslant 7$ by Øbro [15, page 67], but little more is known about it. Both Payne and McDuff [11, 16] remark that it is not even known whether there is a monotone polytope with $\mathcal{E}(P) = \{0\}$.

Nill [13] proposed the following generalization of Conjecture 5 to smooth polytopes:

Conjecture 6 (General Ewald's Conjecture, Nill 2009 [13])**.** *Let $n \in \mathbb{N}$. If P is an n-dimensional smooth lattice polytope in $\mathbb{R}^n$ with the origin in its interior then $\mathcal{E}(P)$ contains a unimodular basis of $\mathbb{Z}^n$.*

This is clearly stronger than Conjecture 5, but it might actually be equivalent; as Nill points out, Conjecture 5 implies that $\mathcal{E}(P)$ linearly spans $\mathbb{R}^n$ for every smooth lattice polytope P with $0 \in \mathrm{Int}(P)$. (The implication is not on a dimension-by-dimension basis).

2 Three Ewald conditions and their motivation in symplectic geometry

Øbro's computational verification of Conjecture 5 for $n \leqslant 7$ shows the following strong version of it: for every facet F of P, $\mathcal{E}(P) \cap F$ contains a unimodular basis. This serves as motivation for the definition we give next. Before that let us introduce the following notation: let P be any polytope and let $\mathcal{F}$ and $\mathcal{R}$ be the sets of facets and *ridges* (that is, faces of codimension two) of P. For a face f of P we denote:

$$\mathrm{Star}(f) = \bigcup_{f \subset F \in \mathcal{F}} F; \quad \mathrm{star}(f) = \bigcup_{f \subset R \in \mathcal{R}} R; \quad \mathrm{Star}^*(f) = \mathrm{Star}(f) \setminus \mathrm{star}(f).$$

Definition 7 (Ewald conditions, McDuff [11, Definition 3.5])**.** *Let P be an n-dimensional polytope with the origin in its interior. We say that:*

1. *P satisfies the* weak Ewald condition *if $\mathcal{E}(P)$ contains a unimodular basis of $\mathbb{Z}^n$.*
2. *P satisfies the* strong Ewald condition *if, for each facet F of P, the set $\mathcal{E}(P) \cap F$ contains a unimodular basis of $\mathbb{Z}^n$.*
3. *A face f of P satisfies the* star Ewald condition *or* is star Ewald *if there exists $\lambda \in \mathcal{E}(P)$ such that $\lambda \in \mathrm{Star}^*(f)$ and $-\lambda \notin \mathrm{Star}(f)$.*
4. *P satisfies the* star Ewald condition *if every face of P satisfies it.*

The star Ewald condition is motivated by the following problem in symplectic toric geometry. It is known that every symplectic toric manifold M has a particular *central* toric orbit that is not *displaceable* by a Hamiltonian isotopy. A relevant question is whether for a given manifold this central orbit is the only non-displaceable one. If this happens then the central orbit is called a *stem.* McDuff relates displaceability of toric orbits in M to *displaceability by probes* of points in the corresponding momentum polytope (a concept that she defines). More precisely, she proves the following:

Theorem 8 (McDuff [11])**.**
1. *Let M be a symplectic toric manifold with momentum polytope P. If a point $u \in \mathrm{Int}(P)$ is displaceable by a probe then its fiber $L_u \subset M$ is displaceable by a Hamiltonian isotopy [11, Lemma 2.4].*
2. *A monotone polytope P satisfies the star Ewald condition if and only if every point of $\mathrm{Int}(P) \setminus \{0\}$ is displaceable by a probe [11, Theorem 1.2].*

It follows that if the momentum polytope of a monotone symplectic toric manifold satisfies the star Ewald condition then the central fiber is a stem.

The star Ewald condition is stronger than the weak Ewald condition by [11, Lemma 3.7]. However, there are 6-dimensional monotone polytopes for which the star Ewald condition fails [11, footnote to p. 134] (see also [4, Proposition 3.11]). Hence, the strong Ewald condition does not imply the star Ewald condition.

3 Deeply smooth polytopes satisfy the Ewald conditions

Definition 9 ([4, Definition 4.9]). *Let v be a vertex of a lattice smooth polytope P in $\mathbb{R}^n$, and let $u_1, \dots, u_n$ be the primitive edge vectors at v. The parallelepiped*

$$\{v + \sum_{i=1}^{n} \lambda_i u_i \mid \lambda_i \in [0, 1] \ \forall i\}$$

is called the corner parallelepiped *of P at v.*

We say that P is deeply smooth *if it contains the corner parallelepiped of P at v for every vertex v of P. We call P* deeply monotone *if it is deeply smooth and monotone.*

Our first main result in [4] determines a class of polytopes for which the Ewald conditions hold:

Theorem 10 ([4, Theorem 4.14]). *Every deeply monotone polytope satisfies the strong and star Ewald conditions (and, consequently, also the weak condition).*

As far as we know this is the first result covering a broad case of Ewald's Conjecture in arbitrary dimension. In the following table we have computed how many monotone polytopes fall within this class for $n \leqslant 6$.

dimension	monotone	deeply monotone
3	18	16
4	124	72
5	866	300
6	7622	1352

4 (Monotone) fiber bundles and neat polytopes

Now we look at the Ewald sets of bundles over polytopes.

Definition 11 (Bundle of a polytope). *Let $n, k \in \mathbb{N}$. Given three polytopes $P \subset \mathbb{R}^{k+n}$, $B \subset \mathbb{R}^k$ and $Q \subset \mathbb{R}^n$, we say that P is a bundle with base B and fiber Q if the following conditions hold:*

1. *P is combinatorially equivalent to $B \times Q$.*
2. *There is a short exact sequence of linear maps*

$$0 \to \mathbb{R}^n \xrightarrow{i} \mathbb{R}^{k+n} \xrightarrow{\pi} \mathbb{R}^k \to 0$$

such that $\pi(P) = B$ and for every $x \in B$ we have that the polytope $Q_x := \pi^{-1}(x) \cap P$ is normally isomorphic to $i(Q)$ (that is, they have the same normal fan).

If P is a monotone bundle with fiber Q and base B then, with the natural identification $Q \cong \{0\} \times Q$, we have that $\mathcal{E}(Q) \subset \mathcal{E}(P)$. It is natural to ask under what conditions we have the analog property for the base: that every point in $\mathcal{E}(B)$ lifts to $\mathcal{E}(P)$. The answer is the following:

Definition 12 (Neat polytope [4, Definition 2.2]). *Let $m, n \in \mathbb{N}$. Let P be a smooth lattice polytope in $\mathbb{R}^n$ defined by the inequalities $Ax \leqslant c$, where $A \in \mathbb{Z}^{m \times n}$ and $c \in \mathbb{Z}^m$. For each $b \in \mathbb{Z}^m$ we define*

$$P_b := \{x \in \mathbb{R}^n : Ax \leqslant c + b\}$$

and call it the deformation of P by b. We say that P is neat *if whenever P_b and P_{-b} are normally isomorphic to (i.e., have the same normal fan as) P for a $b \in \mathbb{Z}^m$ we have that*

$$P_b \cap (-P_{-b}) \cap \mathbb{Z}^n \neq 0;$$

that is, there is an integer point $x \in P_b$ such that $-x \in P_{-b}$.

One of our results in [4] says that the condition above is precisely what is required of the fiber Q for the Ewald properties to be preserved under the fiber bundle operation:

Theorem 13 ([4, Corollary 5.10]). *For a lattice smooth polytope Q the following properties are equivalent:*

1. *Q is neat and satisfies the weak (resp. star) Ewald condition.*
2. *Every lattice smooth bundle P with fiber Q and base $[-1,1]$ satisfies the weak (resp. star) Ewald condition.*
3. *Every lattice smooth bundle P with fiber Q and an arbitrary base B satisfies the weak (resp. star) Ewald condition whenever B satisfies it.*

Corollary 14 ([4, Corollary 2.4]).

- *If Conjecture 5 holds then every monotone polytope is neat.*
- *If Conjecture 6 holds then every lattice smooth polytope is neat.*

5 The number of Ewald points

We now turn to discuss how many Ewald points a monotone polytope can have. It is easy to show that for every monotone n-polytope

$$\mathcal{E}(P) \subset \mathcal{E}([-1,1]^n) = \{-1,0,1\}^n,$$

where the first inclusion should be understood modulo unimodular equivalence. Hence, no monotone n-polytope can have more than 3^n Ewald points. Somewhat surprisingly, the number of Ewald points of the monotone cube is asymptotically attained (modulo a factor proportional to $\sqrt{n}$) by the monotone n-simplex and by any bundle with fiber the monotone simplex and base a segment.

In [4] we computed the number of Ewald points for every monotone polytope up to dimension seven. The minimum number in each dimension is as follows:

n	1	2	3	4	5	6	7
$\min \lvert\mathcal{E}(P)\rvert$	3	7	13	27	59	117	243

These numbers seem to grow exponentially, which supports the claim made in Conjecture 5. In fact, we have an explicit construction of monotone n-polytopes with $|\mathcal{E}(P)|$ growing asymptotically as $3^{2n/3}$ and which achieves *exactly* the minimal size for all $n \in [3,7]$:

Theorem 15 ([4, Corollary 6.7]). *For each $n \geqslant 3$ there is a monotone n-polytope P_n with*

$$|\mathcal{E}(P_n)| = \begin{cases} 13 \cdot 3^{2k-2} & \text{if } n = 3k \\ 3^{2k+1} & \text{if } n = 3k+1 \\ 59 \cdot 3^{2k-2} & \text{if } n = 3k+2 \end{cases}$$

Thus, the minimum number of Ewald points of monotone n-polytopes is of order $O(3^{2n/3})$.

6 Nill's Conjecture: a proof for $n = 2$ and partial results for higher n

In [4] we prove a strong form of Nill's Conjecture in dimension 2, in which the hypothesis is relaxed:

Theorem 16 ([4, Corollary 7.3]). *If P is a lattice polygon with the origin in its interior and each vertex of P is at lattice distance one from the line spanned by its two neighboring boundary lattice points, then $\mathcal{E}(P)$ contains a lattice basis.*

It seems quite challenging to make the type of arguments we use to work in dimensions 3 or higher, but in [4] we were able to prove the following two partial results.

Definition 17 ([4, Definition 7.4]). *Let P be a lattice polytope, F a face of it, and $x_0 \in P$. The maximum distance from x_0 to the facets containing F is called* distance from x_0 to F. *We say that x_0 is* next to F *if it is in the interior of P and at distance one from F.*

Proposition 18 ([4, Proposition 7.5]). *Let P be a deeply smooth n-polytope with the origin in its interior, and suppose that the origin is next to a certain vertex v. Then, $\mathcal{E}(P)$ contains the lattice basis consisting of the primitive edge vectors of P at v.*

Proposition 19 ([4, Proposition 7.7]). *Let P be a smooth 3-polytope with the origin in its interior, and suppose that the origin is next to a certain edge uv. Then, $\mathcal{E}(P)$ contains a lattice basis.*

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