Creating trees with high maximum degree *

Grzegorz Adamski†1, Sylwia Antoniuk‡1, Małgorzata Bednarska-Bzdęga§1, Dennis Clemens ¶2, Fabian Hamann ‖2, and Yanick Mogge **2

[1]Department of Discrete Mathematics, Faculty of Mathematics and CS, Adam Mickiewicz University, Poznań, Poland
[2]Institute of Mathematics, Hamburg University of Technology, Hamburg, Germany.

Abstract

We consider positional games where the winning sets are edge sets of copies of fixed spanning trees or tree universal graphs. We prove that in Maker-Breaker games on the edges of a complete graph K_n, Maker has a strategy to occupy the edges of a graph which contains copies of all spanning trees with almost linear maximum degree, and we give a similar result for Waiter-Client games. By this, it follows that both Maker and Waiter can play at least as good as predicted by the so-called random graph intuition. Moreover, our results improve on special cases of earlier results by Johannsen, Krivelevich, and Samotij as well as Han and Yang. Additionally, when the target of the building player is a copy of only one fixed spanning tree, then we show that in the Waiter-Client game on K_n, Waiter can do even better than suggested by the random graph intuition, while the same is not true for Client in the similarly looking Client-Waiter game.

1 Introduction

Tree embedding problems have a long history, ranging from the embedding of a fixed tree (e.g. [13, 18, 19]) over universality results (e.g. [11, 17, 22]) to packing problems (e.g. [3, 16, 23]). Research in this branch of combinatorics was influenced by many beautiful problems, including the appearance of particular subgraphs in the binomial random graph $G(n, p)$, as well as challenging conjectures, such as the well known Ringel's Conjecture from 1968 and Gyárfás Tree Packing Conjecture from 1978, just to mention a few. For an overview on general graph embedding problems we recommend the survey [6].

In our paper, we want to take a look at such tree embedding problems from a game theoretic perspective, as it has been started already in a series of papers, see e.g. [5, 7, 8, 10, 12, 17, 21]. In general, given any hypergraph $\mathcal{H} = (X, \mathcal{F})$, a *positional game* on $\mathcal{H}$ is played as follows. Two players claim the elements of the *board* X in rounds according to some predefined rule; and the winner is determined according to some rule that involves the *winning sets* in $\mathcal{F}$. Specifically, we will be interested in the following three types of such games.

- **Maker-Breaker** games: Maker and Breaker alternatingly claim one element of X which was not claimed before. Maker wins if she occupies all elements of a winning set, and Breaker wins otherwise.

*The full version of this work can be found in [1, 2] and will be published elsewhere. The research of the fourth and sixth author is supported by Deutsche Forschungsgemeinschaft (Project CL 903/1-1).

†Email: grzegorz.adamski@amu.edu.pl
‡Email: sylwia.antoniuk@amu.edu.pl
§Email: mbed@amu.edu.pl
¶Email: dennis.clemens@tuhh.de
‖Email: fabian.hamann@tuhh.de
**Email: yannick.mogge@tuhh.de

- **Waiter-Client** games: In each round, Waiter offers two elements of X to Client, and then Client decides which element is claimed by him, and which element goes to Waiter. Client wins if he avoids to claim a full winning set, and otherwise Waiter wins. (If in the last round there is only one unclaimed element in X, then it is given to Waiter.)

- **Client-Waiter** games: The elements of X are claimed in the same way as in Waiter-Client games, but this time Client wins if at some point he occupies a winning set, and Waiter wins otherwise.

We note that the above games, when played on the edges of the complete graph K_n, often but not always show to have some strong connection to properties of random graphs, referred to as *random graph intuition*, which roughly speaking suggests that the outcome of a game between perfect players can be predicted by looking at the typical behaviour of a randomly played game in which each player creates a random graph. Prominent examples for such a relation between positional games and random graphs are e.g. the Maker-Breaker clique game [4], the Maker-Breaker Hamiltonicity game [20], and the Waiter-Client H-game [24]. For a general overview on positional games we refer to the monograph [15].

In the following we will stick to games on $X = E(K_n)$, the edge set of a complete graph K_n on n vertices. For any spanning tree T of K_n, we will consider the family $\mathcal{F}_T$ consisting of all copies of T in K_n. Moreover, we will be interested in the family $T(n, \Delta)$ of all graphs which are *universal* for trees on n vertices with maximum degree at most Δ, i.e. graphs which contain a copy of every such tree.

Starting with games in which Maker wants to claim a copy of a fixed tree, Ferber, Hefetz and Krivelevich [10] asked for the largest value $d = d(n)$ such that in a Maker-Breaker game on the edges of K_n, Maker has a strategy to claim a copy of any tree T provided that the maximum degree satisfies $\Delta(T) \leq d$ and n is large enough. An analogue question for Waiter-Client games has then been asked in [8], and related questions regarding tree universality were studied in [5, 17]. We note that in all cases the random graph intuition would suggest that the largest value for the maximum degree $\Delta(T)$ such that the building player (i.e. the player who aims for a winning set) wins should be of the order $\frac{n}{\log(n)}$, see e.g. [19] for the case when a tree T is fixed. However, all previously known results are quite far away from this desired bound on $\Delta(T)$: Hefetz et al. [14] proved that Maker can claim a Hamilton path within $n-1$ rounds. With a tiny worsening in the number of rounds, this was extended to trees of constant maximum degree [7] and trees with $\Delta(T) \leq n^{0.05}$ [10]. Not aiming for a fast winning strategy, Johannsen, Krivelevich, and Samotij [17] further improved the bound on the maximum degree, where their result is much more general as it also considers games played on expander graphs and it gives a Maker's winning strategy for tree universality, i.e. for $T(n, \Delta)$, when $\Delta \leq \frac{cn^{1/3}}{\log(n)}$. Recently, the latter was further improved to $\Delta \leq \frac{cn^{1/2}}{\log(n)}$ by Han and Yang [12]. Moreover, all of the above results stay true when considered in the Waiter-Client context, see [5, 8].

2 Tree Universality

As our first contribution to positional games involving spanning trees, we show that for the tree universality game $T(n, \Delta)$, Maker and Waiter can play at least as good as predicted by the random graph intuition.

Theorem 1 (Tree Universality, Maker-Breaker version, Theorem 1.1 in [2])**.** *There exists a constant $c > 0$ such that the following holds for every large enough integer n. In the Maker-Breaker game on K_n, Maker has a strategy to occupy a graph which contains a copy of every tree T with n vertices and maximum degree $\Delta(T) \leq \frac{cn}{\log(n)}$.*

Theorem 2 (Tree Universality, Waiter-Client version, Theorem 1.2 in [2])**.** *There exists a constant $c > 0$ such that the following holds for every large enough integer n. In the Waiter-Client game on*

K_n, Waiter has a strategy to force Client to claim a graph which contains a copy of every tree T with n vertices and maximum degree $\Delta(T) \leq \frac{cn}{\log(n)}$.

For the proofs of Theorem 1 and Theorem 2 we combine many different tools, including properties of expander graphs, simple absorption and random embedding arguments as well as winning criteria for positional games. While most of our tools are rather standard, the difficulty and novelty in our proof, when compared with the earlier results in [12, 17], lies in finding a suitable list of structural properties which (a) help to embed every tree of the mentioned maximum degree and (b) can be achieved by Maker and Waiter, respectively. Note that the more structural properties are added to such a list, the easier (a) can be proven, but the more difficult (b) gets. The following theorem provides such a list.

Theorem 3 (Theorem 3.1 in [2])**.** *Let $\alpha \in (0, 1)$, and $C_0 > 0$ be any constants. There exist constants $\gamma', c > 0$ and a positive integer n_0 such that the following is true for every $\gamma \in (0, \gamma')$ and every integer $n \geq n_0$.*

Let $G = (V, E)$ be a graph on n vertices with a partition $V = V_1 \cup V_2$ of its vertex set such that the following properties hold:

(1) Partition size: $|V_2| = 500\lfloor \gamma n \rfloor$.

(2) Suitable star: *There are a vertex x^* and disjoint sets $R^*, S^* \subset V_1$ such that the following holds:*

- *(a) $|S^*| = \lfloor 25C_0 \log(n) \rfloor$ and $S^* \subset N_G(x^*)$.*
- *(b) $|R^*| \leq 25$ and for each $v \in R^*$ the following holds: If v is not adjacent with x^*, then v is adjacent with a vertex $s_v \in S^*$, such that $s_v \neq s_w$ if $v \neq w$.*
- *(c) For all $w \in V \setminus (R^* \cup S^*)$, we have $d_G(w, S^*) \geq 2C_0 \log(n)$.*

(3) Pair degree conditions: *For every $v \in V(G)$ there are at most $\log(n)$ vertices $w \in V(G)$ such that $|N_G(v) \cap N_G(w) \cap V_1| < \alpha n$.*

(4) Edges between sets: *Between every two disjoint sets $A \subset V_1$ and $B \subset V$ of size $\lfloor C_0 \log(n) \rfloor$ there is an edge in G.*

(5) Suitable clique factor: *In $G[V_2]$ there is a collection $\mathcal{K}$ of $100\lfloor \gamma n \rfloor$ vertex-disjoint K_5-copies such that the following holds:*

- *(a) There is a partition $\mathcal{K} = \mathcal{K}_{good} \cup \mathcal{K}_{bad}$ such that $|\mathcal{K}_{bad}| = \lfloor \gamma n \rfloor$.*
- *(b) Every vertex $v \in V$ which is not in a clique of $\mathcal{K}_{good}$ satisfies $d_G(v, V_2) \geq 40\lfloor \gamma n \rfloor$.*
- *(c) For every clique $K \in \mathcal{K}_{good}$ there are at most γn cliques $K' \in \mathcal{K}_{good}$ such that G does not have a matching of size 3 between $V(K)$ and $V(K')$.*

Then G contains a copy of every tree T on n vertices with maximum degree $\Delta(T) \leq \frac{cn}{\log(n)}$.

The proof of Theorem 3 can be found in [2], and its overall idea can be summarized as follows. We make a case distinction depending on whether the given tree T contains many bare paths of suitable length (i.e. paths such that all inner vertices have degree 2 in the given tree) or many leaves. In the first case, we embed T minus the bare paths into V_1, by using the properties (3) and (4) together with a criterion by Haxell [13] that helps to embed almost spanning trees into expander graphs. Then, with property (5), we manage to embed all the remaining bare paths to complete a copy of T, and at the same time absorb all leftover vertices from V_1 into our embedding. In the second case, we proceed similarly and embed the leaves at the end of our embedding procedure. However, in order to succeed with this final embedding step, we slightly modify the first step involving Haxell's criterion as follows: If there is a vertex x in T which is adjacent to many neighbours of leaves, we modify Haxell's criterion to make sure that x can be embedded onto x^* (see property (2)) and that we can use S^* exclusively for the embedding of leaf neighbours. Otherwise, if such a vertex x does not exist, we make sure that in the application of Haxell's criterion a small subtree of T, which itself contains many leaf neighbours, is embedded in a suitable (i.e. random) way into V_1. In both cases, also using the properties (2)–(4), we then obtain suitable properties for our partial embedding that help to finish the embedding of T with a generalization of Hall's Theorem.

For the final proofs of Theorem 1 and Theorem 2, i.e. for giving strategies that create a graph satisfying the properties (1)–(5), we combine many standard tools from positional games, including results on degree games, clique factor games plus the well-known Erdős-Selfridge Criterion and variants of it. A novelty in our proof is that we also play a pair degree game which is necessary for applying our random embedding argument above. While for Maker-Breaker games property (3) cannot be improved in the sense that each pair of vertices gets a large common neighbourhood, for Waiter-Client games we can prove the following more general statement which allows to obtain large common neighbourhoods for all sets of at most logarithmic size.

Lemma 4 (Lemma 6.2 in [1]). *Let $\beta \in (0,1)$. Then for every large enough integer n and every $t \in \mathbb{N}$ such that $t \leq 0.1 \log_2(n)$ the following holds. Suppose G is a graph on n vertices and for every set A of t vertices we have a set $Y_A \subset N_G[A]$ of at least βn common neighbours. Then in the Waiter-Client game on G, Waiter has a strategy such that at the end of the game, Client's graph C satisfies the following:*

$$|N_C[A] \cap Y_A| \geq \frac{\beta n}{200^{t+1}} \quad \text{for every } A \subset V(G) \text{ such that } |A| = t.$$

We believe that the above lemma could be of independent interest, as it may be helpful for other games in which Waiter's goal is to claim complex spanning structures.

3 Results on fixed trees

We believe that the bound of $\frac{n}{\log n}$ in Theorem 1 is best possible and pose this as a conjecture. One reason for believing in this conjecture is that Maker-Breaker games often behave as predicted by the random graph intuition, or Maker performs even worse than this prediction. Indeed, for the randomly played game it follows from [18] that there are fixed trees of maximum degree $\Theta(\frac{n}{\log n})$ which with high probability are not contained in Maker's random graph.

For Waiter-Client games, the situation is completely different, and in fact, we can prove that for any fixed tree T of not too large but linear maximum degree, Waiter has a winning strategy for the Waiter-Client game with winning sets $\mathcal{F}_T$. It then becomes natural to ask for the largest constant c such that Waiter can always win if $\Delta(T) \leq cn$ and n is large enough. With the following two theorems we give a small window for the size of c.

Theorem 5 (Theorem 1.2 in [1]). *For every $\varepsilon \in \left(0, \frac{1}{3}\right)$ there exist positive constants b and n_0 such that the following holds. Let T_n be a tree on $n \geq n_0$ vertices with $\Delta(T_n) < \left(\frac{1}{3} - \varepsilon\right) n$. Then Waiter has a strategy to force a copy of T_n in the Waiter-Client game on K_n within at most $n + b\sqrt{n}$ rounds.*

Theorem 6 (Theorem 1.3 in [1]). *There are positive constants γ and n_0 such that the following holds for every $n \geq n_0$. There exists a tree T_n with n vertices and $\Delta(T_n) < \left(\frac{1}{2} - \gamma\right) n$ such that Client can avoid claiming a copy of T_n in the Waiter-Client game on K_n.*

The proof of Theorem 5, which is given in [1], is an involved study of ad-hoc winning strategies for Waiter consisting of several cases and stages, depending on the existence and distribution of large degree vertices in T, the structure of the tree after all such vertices get deleted, and the existence of suitable bare paths as well as matchings incident with leaves. We skip the details here.

In contrast to this, Theorem 6 is obtained by analysing a partially randomized strategy for Client. We prove this theorem with $\gamma = 0.001$ but do no effort to optimize it, as we believe that our randomized strategy is not optimal. We also note that it is easy to find trees with maximum degree close to $\frac{n}{2}$ that

Client can avoid. Although this improvement by the constant γ in Theorem 6 may seem cosmetic, we believe that it is important for determining a best possible constant c for which Waiter can always win if $\Delta(T) \leq cn$ and n is large enough.

Finally, we consider the Client-Waiter version of the above game. In contrast to the above results, it turns out that Client, who is the building player now, cannot do better than predicted by the random graph intuition. Indeed, the following statement can be obtained as a corollary of Lemma 4.

Theorem 7 (Theorem 1.3 in [1]). *There are positive constants c and n_0 such that the following holds. For every $n \geq n_0$ there exists a tree T_n with n vertices and $\Delta(T_n) \leq \frac{cn}{\log(n)}$ such that in a Client-Waiter game on K_n, Waiter can prevent Client from claiming a copy of T_n.*

4 Open problems

As already stated, we believe that Theorem 1 is optimal up to the constant factor c, but we think that Waiter can do better. Therefore, we state the following two conjectures.

Conjecture 8. *There exists a constant $C > 0$ such that the following holds for every large enough integer n. In the Maker-Breaker game on K_n, Breaker has a strategy such that Maker cannot build a graph which contains a copy of every tree T with n vertices and maximum degree $\Delta(T) \leq \frac{Cn}{\log(n)}$.*

Conjecture 9. *There exists a constant $c > 0$ such that the following holds for every large enough integer n. In the Waiter-Client game on K_n, Waiter has a strategy to force Client to claim a graph which contains a copy of every tree T with n vertices and maximum degree $\Delta(T) \leq cn$.*

Similarly and based on other known results on Client-Waiter games we suspect that the Client-Waiter game with winning sets $T(n, \Delta)$ behaves according to the random graph intuition. Due to Theorem 7 it remains to prove the following conjecture.

Conjecture 10. *There exists a constant $c > 0$ such that the following holds for every large enough integer n. In the Client-Waiter game on K_n, Client has a strategy to build a graph which contains a copy of every tree T with n vertices and maximum degree $\Delta(T) \leq \frac{cn}{\log(n)}$.*

Last but not least, recall that in our strategy for Theorem 2 it was beneficial to know that Waiter can force a spanning graph where every pair of vertices has a common neighbourhood of linear size. We wonder how large this pair degree can be made.

Problem 11. *Find the maximum α such that for every large enough n Waiter has a strategy in the Waiter-Client game on K_n to force Client to claim a spanning subgraph C with the following property: for any two vertices v, w we have $|N_C(v) \cap N_C(w)| \geq \alpha n$.*

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