Speed and size of dominating sets in domination games*

Ali Deniz Bagdas[†1], Dennis Clemens[‡1], Fabian Hamann[§1], and Yannick Mogge[¶1]

[1]Institute of Mathematics, Hamburg University of Technology, Hamburg, Germany

Abstract

We consider Maker-Breaker domination games, a variety of positional games, in which two players (Dominator and Staller) alternately claim vertices of a given graph. Dominator's goal is to fully claim all vertices of a dominating set, while Staller tries to prevent Dominator from doing so, or at least tries to delay Dominator's win for as long as possible.

We prove a variety of results about domination games, including the number of turns Dominator needs to win and the size of a smallest dominating set that Dominator can occupy, when considering e.g. random graphs, powers of paths, and trees. We could also show that speed and size can be far apart, and we prove further non-intuitive statements about the general behaviour of such games.

We also consider the Waiter-Client version of such games.

1 Introduction

Let a hypergraph $\mathcal{H} = (X, \mathcal{F})$ and two integers $m, b \geq 1$ be given. The $(m : b)$ *Maker-Breaker game* on $(X, \mathcal{F})$ is played as follows. Maker and Breaker alternate in moves, where in a move Maker claims up to m unclaimed elements of the *board* X, and Breaker claims up to b unclaimed elements of X. Maker wins if during the course of the game she manages to claim all elements of a *winning set*, i.e. a hyperedge from $\mathcal{F}$, while Breaker wins otherwise. Surely, this outcome can depend on who makes the first move. Therefore, whenever it makes a difference in the following, we will state clearly whom we assume to be the first player. If $m = b = 1$, the game is called *unbiased*; and otherwise it is called *biased*. For a nice overview about positional games in general we recommend the monograph [15] as well as the survey [18].

Let $G = (V, E)$ be a graph. We denote the set of vertices of G by $V(G)$ and let $v(G) = |V(G)|$. We will mostly focus on *Maker-Breaker domination games*, a certain variety of Maker-Breaker games, which were recently introduced by Duchêne et al. [9]. While most Maker-Breaker games are played on the edge set of some graph, domination games are played on the vertex set of a given graph G instead. Two players, who are called Dominator and Staller, alternately claim vertices of G, and Dominator (who is playing as Maker) wins if and only if she manages to occupy all vertices of a *dominating set*, which is a subset of $V(G)$ such that every $v \in V(G)$ is either a neighbour of the dominating set or part of this subset itself. Note that the renaming of the players Maker and Breaker as Dominator and Staller is done to be consistent with the usual domination games; for an overview on these games we recommend the book [4].

*The full version of this work will be published elsewhere. This research of the second and fourth author is supported by Deutsche Forschungsgemeinschaft (Project CL 903/1-1).

†Email: ali.bagdas@tuhh.de

‡Email: dennis.clemens@tuhh.de

§Email: fabian.hamann@tuhh.de

¶Email: yannick.mogge@tuhh.de

Let $\gamma_{MB}(G, m : b)$ denote the smallest number of rounds in which Dominator can always win the $(m : b)$ Maker-Breaker domination game on G, provided that she starts the game, and where we set $\gamma_{MB}(G, m : b) = \infty$ if Dominator does not have a winning strategy. Similarly, let $\gamma'_{MB}(G, m : b)$ denote the smallest number of rounds for the case when Staller starts the game, and for short, let $\gamma_{MB}(G) := \gamma_{MB}(G, 1 : 1)$ and $\gamma'_{MB}(G) := \gamma'_{MB}(G, 1 : 1)$.

2 Known results

In their paper which introduced Maker-Breaker domination games, Duchêne et al. [9] proved that deciding who wins an unbiased Maker-Breaker domination game is PSPACE-complete. On the other hand, Gledel et al. [13] put a focus on the number of rounds which Dominator needs to win, and determined $\gamma_{MB}(G)$ and $\gamma'_{MB}(G)$ precisely when G is a tree or a cycle. Partial results for Cartesian products and paths [8, 11] as well as Corona products [7] of graphs were obtained afterwards as well. Additionally, Gledel et al. [13] provided examples which show that the domination number $\gamma(G)$ of a graph G and the Maker-Breaker domination numbers $\gamma_{MB}(G)$ and $\gamma'_{MB}(G)$ can take arbitrary values with the obvious restriction that $\gamma(G) \leq \gamma_{MB}(G) \leq \gamma'_{MB}(G)$. In particular, all these three values can be arbitrarily far apart from each other.

Theorem 1 (Theorem 3.1 in [13]). *For any integers $2 \leq r \leq s \leq t$, there exists a graph G such that $\gamma(G) = r$, $\gamma_{MB}(G) = s$ and $\gamma'_{MB}(G) = t$.*

3 Our results I: unbiased games

First we are interested in the general behaviour of domination games in the unbiased setting and, in particular, to find necessary or sufficient conditions for Dominator to win in given time. In this regards, Gledel et al. [13] already proved the following proposition. For this, for a graph G, let $X_\gamma(G)$ denote the number of dominating sets of size γ of G.

Proposition 2 (Proposition 3.3 in [13]). *If G is a graph and $X_\gamma(G) < 2^{\gamma(G)-1}$, then $\gamma_{MB}(G) > \gamma(G)$.*

Instead of looking at the number of smallest possible dominating sets, we proved the following minimum degree condition for Dominator to have a winning strategy.

Theorem 3. *Let n be a positive integer and let $\delta(G)$ denote the minimum degree of G. If G is a graph on n vertices with $\delta(G) > \log_2(n) - 1$, then Dominator wins the $(1 : 1)$ Maker-Breaker domination game on G.*

Moreover, the bound on $\delta(G)$ is asymptotically best possible. For infinitely many n, there is a graph G on n vertices and with $\delta(G) > \log_2(n) - 2$ such that Staller wins the $(1 : 1)$ Maker-Breaker domination game on G.

Next to this, when asking for the existence of winning strategies, it seems natural to study the behaviour of the Maker-Breaker domination game when played on a randomly chosen graph. Let $G \sim G_{n,p}$ denote a graph sampled from the binomial random graph model, where each edge of a graph with n vertices is present with probability p. When we play a Maker-Breaker domination game on such a graph with constant probability p, we have the following bound on the number of turns that Dominator needs to win.

Theorem 4. *If $p \in (0, 1)$ is constant and $G \sim G_{n,p}$, then a.a.s.*

$$\gamma_{MB}(G) = (1 + o(1)) \log_{1/(1-p)}(n).$$

Although the proofs of both Theorem 3 and Theorem 4 can be done with fairly standard methods from positional games theory, we believe that these statements are important for getting a general intuition for domination games and for predicting the outcome of such games.

4 Our results II: biased games

A lot of research in positional games considers games with a bias, yet we do not know of any paper considering biased versions of Maker-breaker domination games. As a first step, we extend [13] by proving results for biased game in which Dominator wants to dominate all vertices of the power of any path, or all vertices of a tree. Let P_n^k denote the k-th power of a path with n vertices. Then the following theorem holds which can be proven with an inductive argument that mainly involves ad-hoc winning strategies with case distinctions.

Theorem 5. *For all integers $b, k \leq n$ it holds that*

$$\gamma_{MB}(P_n^k, b:1) = \left\lceil \frac{n-1}{b(2k+1)-1} \right\rceil .$$

For a given a tree T and a bias b, let us say that T is b-good if we can recursively delete vertices, which have exactly b leaf neighbours, and also delete these leaf neighbours, until we reach a forest where every vertex has at most $b-1$ leaf neighbours. Then the following holds.

Theorem 6. *Let T be a tree with $v(T) \geq 2$. Then the following are equivalent:*

(i) Dominator wins the $(b:1)$ game on T when Staller is the first player.

(ii) T is b-good.

While the proof of the implication (i)⇒(ii) is a simple exercise, the other direction is less trivial. Here, we do an induction for a slightly stronger statement which considers games in which Dominator's goal is to dominate only a certain subset of vertices of T. Moreover, by having this more general statement we are also able to give an analogue theorem for the case when Dominator is the first player. We skip the details here, and will soon make them available on arXiv.

Additionally, in the case that a given tree T is not b-good, we are still able to prove the following quantitative statement.

Theorem 7. *For every tree T it holds that Dominator can dominate at least $\left(1 - \frac{1}{(b+1)^2}\right) v(T)$ vertices in the $(b:1)$ game on T. Moreover, the bound is sharp.*

Note that for the above games involving trees, we do not consider Staller's bias to be larger than 1 due to the fact that with bias 2, Staller can already win the game within one round, by claiming a leaf and its neighbour. Still, for other graphs, it makes sense to increase Staller's bias and in fact, interesting (and maybe surprising) behaviours can be shown, see Theorem 8.

Before stating this theorem, note that a nice property of $(m:b)$ Maker-Breaker games is that these are monotone with respect to each of the biases m and b. That is, roughly speaking, increasing the bias of one of the players can never be a disadvantage for this player; see e.g. [2, 15]. This observation leads to the natural definition of *threshold biases*, which in many cases have proven to be related to properties of random graphs, see e.g. [12, 17]. When both biases get increased simultaneously, we however cannot expect monotonicity in general. For an example, Balogh et al. [1] considered the 2-diameter game in which Maker's goal is to occupy a spanning subgraph of K_n with diameter 2, and they proved that Breaker wins the $(1:1)$ variant of this game, while Maker has a winning strategy for the $(2:b)$ variant even when $b \leq \frac{1}{9} n^{1/8} (\log n)^{-3/8}$, provided n is large enough. In particular, the $(b:b)$ variant is won by Maker for every constant $b \geq 2$ if n is large.

Now, looking only at these fair $(b:b)$ games, the above example could still be considered to be monotone for all $b \geq 1$, since increasing b never worsens Maker's chances of winning. So, one could wonder whether such a behaviour always holds for fair Maker-Breaker games. With our next result we show that this is not the case, even for Maker-Breaker domination games.

Theorem 8. *Let $B \subset \mathbb{N}$ be any finite set. Then there exists a graph G, such that Dominator wins the $(b:b)$ Maker-Breaker domination game on G (when Dominator starts) if and only if $b \notin B$.*

5 Our results III: speed and size

Another interesting question is the following: when playing the $(m : b)$ Maker-Breaker domination game on a graph G, of what size is the smallest dominating set which Dominator can always claim? Let $s_{MB}(G, m : b)$ and $s_{MB}(G) = s_{MB}(G, 1 : 1)$ denote this values when Dominator starts the game, and let $s'_{MB}(G, m : b)$ and $s'_{MB}(G) = s_{MB}(G, 1 : 1)$ denote this value when Staller starts, where again we set such a value to ∞ if Dominator does not have a winning strategy. A priori it is not clear why the minimal number of rounds and the minimal size of a dominating set that Dominator can achieve should be different. In fact, from the proofs in [13] it can be deduced easily that $\gamma_{MB}(G) = s_{MB}(G)$ and $\gamma'_{MB}(G) = s'_{MB}(G)$ hold when G is a tree or a cycle. However, in contrast to this, we can prove the following statement which, roughly speaking, says that all these parameters in questions can take almost arbitrary values and hence can be arbitrarily far apart. Note that this statement is a strengthening of Theorem 1 from [13].

Theorem 9. *For any biases $m \leq b$ and any integers r, s, s', t, t' such that $m+1 \leq r, \max\{2m+1, r\} \leq s \leq s', t \leq t'$, $s \leq m \cdot t$, $s' \leq m \cdot t'$, there exists a graph G such that*

$$\begin{aligned} &\gamma(G) = r, \\ &s_{MB}(G, m : b) = s \quad \text{and} \quad \gamma_{MB}(G, m : b) = t \\ &s'_{MB}(G, m : b) = s' \quad \text{and} \quad \gamma'_{MB}(G, m : b) = t'. \end{aligned}$$

One step in the proof of this theorem is to provide a construction which allows us to transfer constructive results from general Maker-Breaker games to domination games. We can also use this transference construction to prove the following rather non-intuitive result:

Theorem 10. *For any biases m, b with $m \leq b$ and any integers $t > s \geq 2m + 1$, there exists a graph G such that $s_{MB}(G : m : b) = s$, but Dominator cannot occupy a dominating set of size s before she has occupied another minimal dominating set of size t.*

Moreover, with similar arguments, we can show the following result which roughly states that claiming a smallest possible dominating set can take Dominator arbitrarily much longer than claiming an arbitrarily large dominating set which can be claimed in optimal time. Hence, as already supported by Theorem 9, studying the parameters s_{MB} and γ_{MB} can be two very different problems which may require very different tools when proving exact results.

Theorem 11. *For any biases m, b with $m \leq b$ and any integers $t' \geq t \geq s' \geq s \geq 2m + 1$, there exists a graph G such that $\gamma_{MB}(G, m : b) = t$ and $s_{MB}(G, m : b) = s$, but in the $(m : b)$ Maker-Breaker domination game on G we have that*

- *s' is the smallest size of a dominating set that Dominator can get within t rounds,*
- *t' is smallest number of rounds that Dominator needs to claim a dominating set of size s.*

6 Our results IV: Waiter-Client domination games

Another variety of Maker-Breaker games are *Waiter-Client games* (earlier called Picker-Chooser games, see e.g. [2]), which have received increasing attention lately, ranging from results on fast winning strategies [5, 10] over biased games [3, 14, 19] to games played on random graphs [6, 16]. In the following we will stick to the case of unbiased Waiter-Client games. On a given hypergraph $\mathcal{H} = (X, \mathcal{F})$, these games are played almost the same way as Maker-Breaker games with the following difference: In every round, Waiter chooses two unclaimed elements of the board X and then Client decides which of these elements goes to Waiter while the other one goes to Client. Waiter wins if and only if she manages to claim all elements of a winning set from $\mathcal{F}$.

So far, domination games have not been studied in this setting. So, we also aim to give first results for Waiter-Client domination games and on the relation of Waiter-Client and Maker-Breaker domination games. Given a graph G, we define the *Waiter-Client domination game* on G in the obvious way: Dominator (playing as Waiter) offers two unclaimed vertices of G and then Staller (playing as Client) picks one of these vertices for himself and the other goes to Dominator. In accordance with previous notation, we denote with $\gamma_{WC}(G)$ the smallest number of rounds in which Dominator can always occupy a dominating set in the Waiter-Client domination game on G, and we let $s_{WC}(G)$ denote the size of the smallest dominating set that Waiter can always claim. For our first results in this setup, we can prove that for cycles and trees the game behaves the same way as in the Maker-Breaker setting [13] (when Breaker starts).

Theorem 12. *For every $n \geq 3$,*

$$\gamma_{WC}(C_n) = s_{WC}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor.$$

Theorem 13. *Let T be a tree on n vertices. If T has a perfect matching, then*

$$\gamma_{WC}(T) = s_{WC}(T) = \frac{n}{2}.$$

In all other cases, Dominator does not win the $(1:1)$ Waiter-Client domination game on T.

Due to these results and due to the fact that in the literature, Waiter most of the time can play at least as good as Maker can do in the analogue game with same winning sets, it seems natural to wonder whether relations such as $\gamma_{WC}(G) \leq \gamma_{MB}(G)$ can be proven for arbitrary graphs G. As our last result we negate this with the following theorem which states that the parameters $\gamma_{WC}(G)$ and $\gamma_{MB}(G)$ can take almost arbitrary values and, in particular, $\gamma_{WC}(G)$ can be much larger than $\gamma_{MB}(G)$. The proof again uses our transference argument from the previous section together with suitable hypergraphs on which either Maker (in the usual Maker-Breaker game) or Waiter (in the usual Waiter-Client game) can win fast, while the other player does not have a strategy that ensures a fast win.

Theorem 14. *For all integers $s, t \geq 7$ there is a graph G such that $\gamma'_{MB}(G) = s$ and $\gamma_{WC}(G) = t$.*

References

[1] J. Balogh, R. Martin, and A. Pluhár, The diameter game, *Random Structures & Algorithms* **35.3** (2009), 369–389.

[2] J. Beck, *Combinatorial games: Tic-Tac-Toe theory*, Volume 114, Cambridge University Press, Cambridge (2008).

[3] M. Bednarska-Bzdega, D. Hefetz, M. Krivelevich, and T. Łuczak, Manipulative waiters with probabilistic intuition, *Combinatorics, Probability and Computing* **25.6** (2016), 823–849.

[4] B. Brešar, M. A. Henning, S. Klavžar, and D. F. Rall, *Domination games played on graphs*, Springer, Cham (2021).

[5] D. Clemens, P. Gupta, F. Hamann, A. Haupt, M. Mikalački, and Y. Mogge, Fast strategies in Waiter-Client games, *The Electronic Journal of Combinatorics* **27.3** (2020), 1–35.

[6] D. Clemens, F. Hamann, Y. Mogge, and O. Parczyk, Waiter-Client Games on Randomly Perturbed Graphs, in: *Extended Abstracts EuroComb 2021: European Conference on Combinatorics, Graph Theory and Applications*, Springer (2021), 397–403.

[7] A. Divakaran, T. James, S. Klavžar, and L. S. Nair, Maker–Breaker domination game played on Corona products of graphs, preprint (2024), `arXiv:2402.13581`.

[8] P. Dokyeesun, Maker–Breaker domination game on Cartesian products of graphs, preprint (2023), `arXiv:2310.04103`.

[9] E. Duchêne, V. Gledel, A. Parreau, and G. Renault, Maker–Breaker domination game, *Discrete Mathematics* **343.9** (2020), 111955.

[10] V. Dvořák, Waiter–Client triangle-factor game on the edges of the complete graph, *European Journal of Combinatorics* **96** (2021), 103356.

[11] J. Forcan and J. Qi, Maker–Breaker domination number for Cartesian products of path graphs P_2 and P_n, preprint (2020), `arXiv:2004.13126`.

[12] H. Gebauer and T. Szabó, Asymptotic random graph intuition for the biased connectivity game, *Random Structures & Algorithms* **35.4** (2009), 431–443.

[13] V. Gledel, V. Iršič, and S. Klavžar, Maker–Breaker domination number, *Bulletin of the Malaysian Mathematical Sciences Society* **42** (2019), 1773–1789.

[14] D. Hefetz, M. Krivelevich, and W. E. Tan, Waiter-Client and Client-Waiter planarity, colorability and minor games, *Discrete Mathematics* **339** (2016), 1525–1536.

[15] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó, *Positional games*, Volume 44, Birkhäuser, Basel (2014).

[16] D. Hefetz, M. Krivelevich, and W. E. Tan, Waiter–Client and Client–Waiter Hamiltonicity games on random graphs, *European Journal of Combinatorics* **63** (2017), 26–43.

[17] M. Krivelevich, The critical bias for the Hamiltonicity game is $(1 + o(1))n/\ln n$, *Journal of the American Mathematical Society* **24.1** (2011), 125–131.

[18] M. Krivelevich, Positional games, preprint (2014), `arXiv:1404.2731`.

[19] R. Nenadov, Probabilistic intuition holds for a class of small subgraph games, *Proceedings of the American Mathematical Society* **151.04** (2023), 1495–1501.