Complexity measures of trilean functions *

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Abstract

The study of the relations between different complexity measures of boolean functions led Nisan and Szegedy to state the sensitivity conjecture in 1994. This problem remained unsolved until 2019, when Huang proved the conjecture by means of an equivalent reformulation of the problem in graph theory. We wonder if the same type of results hold for functions defined on finite alphabets of cardinality greater than two. This work concerns functions over an alphabet with three symbols, which we call *trilean functions*. In this context we follow the steps of Nisan and Szegedy and extend most of the results for boolean functions. Also, we find an equivalent reformulation in graph theoretical terms of the trilean version of the sensitivity conjecture.

1 Introduction

One can measure how complex a given boolean function $f : \{0, 1\}^n \to \{0, 1\}$ is in many ways, and these different conceptions give rise to different complexity measures such as: the degree, the sensitivity, the block sensitivity, the decision tree complexity... In 1994, it was already known that all these complexity measures were polynomially related except the sensitivity. This led Nisan and Szegedy to the statement of the sensitivity conjecture [5], which claimed that the sensitivity was also polynomially related to the other measures. This conjecture was proved almost 30 years later by Huang [4], and his proof relies on an equivalence theorem due to Gotsman and Linial [3] which translated the sensitivity conjecture to an equivalent problem in graph theory. An extended summary of this story can be found in [1].

After this study, it is natural to ask whether there is a similar situation with functions $f : T^n \to T$, where T is a finite set of cardinality $m > 2$. Our work focuses on the case $m = 3$ and its corresponding functions, which we call *trilean functions*. For technical reasons, our set T will be the set $\{1, \varepsilon, \varepsilon^2\}$ where ε is a primitive cubic root of unity.

This abstract is divided in four sections. In the first one, we define two complexity measures of a trilean function f, namely the sensitivity $s(f)$ and the degree $\deg(f)$. In Proposition 4 we provide a quadratic upper bound for the sensitivity in terms of the degree. This was also known for boolean functions before 1994. Again, the most difficult question seems to be the construction of a polynomial bound in the other direction. In the second section we prove in Theorem 6 that providing an upper bound for the degree in terms of the sensitivity is equivalent to a graph theoretical problem that can be entirely stated in terms of a generalization of the n-dimensional hypercube. In the third section, by means of Theorem 6 we show that there are trilean functions such that $s(f) < \sqrt{2\deg(f)} + 1$. We conclude with a future work section.

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2 Two complexity measures and the first relation between them

First of all, let's define two complexity measures of trilean functions which generalize those of boolean functions. The first one is the degree, whose definition relies on the following result:

Proposition 1. *Let $f : \{1, \varepsilon, \varepsilon^2\}^n \to \{1, \varepsilon, \varepsilon^2\}$ be a trilean function. Then, there is a unique polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$ of degree at most 2 in each variable that represents f (i.e., $F(\mathbf{x}) = f(\mathbf{x})$ at every $\mathbf{x} \in \{1, \varepsilon, \varepsilon^2\}^n$).*

Definition 2. *The* degree *of a trilean function $f : \{1, \varepsilon, \varepsilon^2\}^n \to \{1, \varepsilon, \varepsilon^2\}$ is the degree of the unique polynomial of degree at most 2 in each variable that represents f.*

The second complexity measure of trilean functions we are considering is the sensitivity.

Definition 3. *The* local sensitivity *of a trilean function f at $\mathbf{x} \in \{1, \varepsilon, \varepsilon^2\}^n$, $s_{\mathbf{x}}(f)$, is the number of elements $\mathbf{y} \in \{1, \varepsilon, \varepsilon^2\}^n$ which differ from $\mathbf{x}$ in exactly one entry and $f(\mathbf{x}) \neq f(\mathbf{y})$.*

The sensitivity *of f is $s(f) = \max_{\mathbf{x} \in \{1, \varepsilon, \varepsilon^2\}^n} \{s_{\mathbf{x}}(f)\}$.*

We have the following polynomial upper bound for the sensitivity of a trilean function in terms of its degree:

Proposition 4. *For every trilean function $f : \{1, \varepsilon, \varepsilon^2\}^n \to \{1, \varepsilon, \varepsilon^2\}$, $s(f) \leq 64 \deg(f)^2$.*

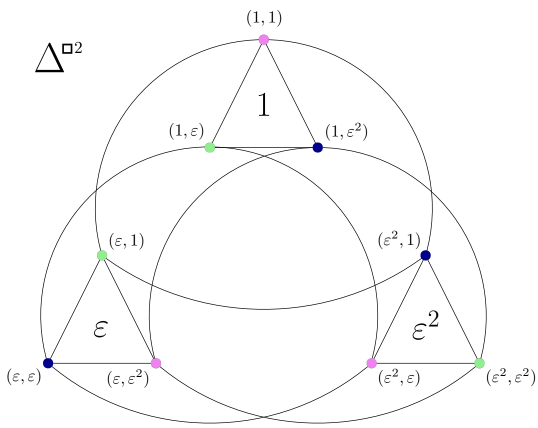

Figure 1: A proper 3-coloring of $\Delta^{\square 2}$.

3 The equivalence theorem

The proof of the sensitivity conjecture for boolean functions would not have been possible without the equivalence theorem by Gotsman and Linial [3], which stated that solving the sensitivity conjecture was equivalent to solving a combinatorial problem in graph theory. That problem consisted on finding a polynomial lower bound on n for the maximum between the maximum degree of every induced subgraph of the n-dimensional hypercube with not exactly half of its vertices and the maximum degree of its complementary.

We have obtained an equivalence theorem in the trilean case, which is somehow more involved than the boolean one. Before presenting its statement, we need to introduce some definitions and notation.

In the case of boolean functions, the set $\{0, 1\}^n$ where they are defined can be seen as the vertex set of the n-dimensional hypercube Q_n. In the context of trilean functions, a natural generalization is the graph G whose vertex set is $\{1, \varepsilon, \varepsilon^2\}^n$ and where two vertices are adjacent if and only if they differ in exactly one of their entries. This graph can be described in terms of the cartesian product of graphs:

Definition 5. *Given two graphs G and H, its* cartesian product *$G \square H$ is the graph with vertex set $V(G) \times V(H)$ and whose edges are the pairs $\{(u,v),(u',v')\}$ such that*

- *$u = u'$ and $\{v, v'\}$ is an edge of H, or*
- *$v = v'$ and $\{u, u'\}$ is an edge of G.*

If we denote the complete graph on three vertices (the triangle graph) by Δ, then the generalization of the n-dimensional hypercube in the above sense is the graph $\Delta \square \overset{n)}{\cdots} \square \Delta = \Delta^{\square n}$. We observe that the product of the entries of every vertex provides a proper 3-coloring of $\Delta^{\square n}$ (see Figure 1 for a 3-coloring of $\Delta^{\square 2}$).

We are going to denote each set of the resulting tripartition by C_i with $i \in \{1, \varepsilon, \varepsilon^2\}$; i.e.,

$$C_i = \left\{ \mathbf{x} = (x_1, \ldots, x_n) \in \{1, \varepsilon, \varepsilon^2\}^n \;\middle|\; \prod_{j=1}^{n} x_j = i \right\} .$$

Furthermore, given three induced subgraphs $H_1, H_\varepsilon, H_{\varepsilon^2}$ of $\Delta^{\square n}$ whose vertex sets constitute a tripartition of $\{1, \varepsilon, \varepsilon^2\}^n$, we define three new induced subgraphs (see Figure 2):

- H'_1, with $V(H'_1) = (V(H_1) \cap C_1) \cup (V(H_{\varepsilon^2}) \cap C_\varepsilon) \cup (V(H_\varepsilon) \cap C_{\varepsilon^2})$.
- H'_ε, with $V(H'_\varepsilon) = (V(H_\varepsilon) \cap C_1) \cup (V(H_1) \cap C_\varepsilon) \cup (V(H_{\varepsilon^2}) \cap C_{\varepsilon^2})$.
- H'_{ε^2}, with $V(H'_{\varepsilon^2}) = (V(H_{\varepsilon^2}) \cap C_1) \cup (V(H_\varepsilon) \cap C_\varepsilon) \cup (V(H_1) \cap C_{\varepsilon^2})$.

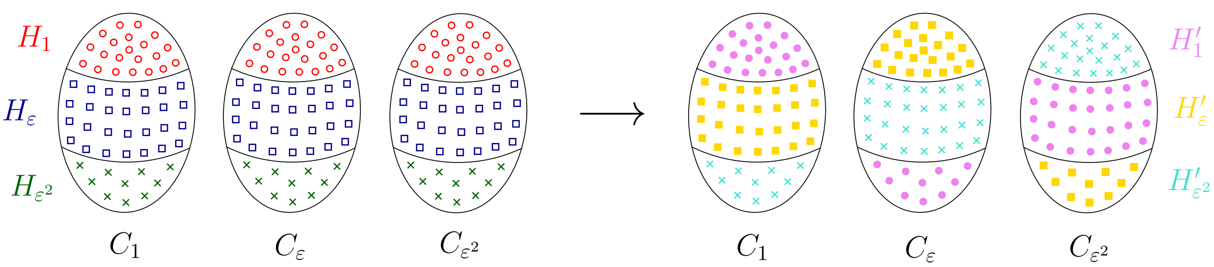

Figure 2: The vertices of $H'_1, H'_\varepsilon, H'_{\varepsilon^2}$ can be visualized as a certain cyclic rotation of the vertices of $H_1, H_\varepsilon, H_{\varepsilon^2}$.

Theorem 6 (Equivalence theorem)**.** *The following are equivalent for any function $h : \mathbb{N} \to \mathbb{R}$:*

1. *For any induced subgraphs H_1, H_ε and H_{ε^2} of $\Delta^{\square n}$ such that their vertex sets constitute a tripartition of $V(\Delta^{\square n})$ and the corresponding H'_1, H'_ε and H'_{ε^2} do not verify that $|V(H'_1)| = |V(H'_\varepsilon)| = |V(H'_{\varepsilon^2})| = 3^{n-1}$,*

$$\Gamma(H_1, H_\varepsilon, H_{\varepsilon^2}) = 2n - \min\{\delta(H_1), \delta(H_\varepsilon), \delta(H_{\varepsilon^2})\} \geq h(n) \ ,$$

where $\delta(H_i)$ is the minimum degree of H_i for all $i \in \{1, \varepsilon, \varepsilon^2\}$.

2. *For any trilean function $f : \{1, \varepsilon, \varepsilon^2\}^n \to \{1, \varepsilon, \varepsilon^2\}$, $s(f) \geq h\left(\frac{1}{2}\deg(f)\right)$.*

Hence, if we could prove the first statement for a polynomial function h, we would have proved the sensitivity conjecture in the trilean case.

4 Searching for trilean functions with low sensitivity and high degree

In the case of boolean functions, Chung, Füredi, Graham and Seymour [2] proved that there exists an induced subgraph of the n-dimensional hypercube with $2^{n-1}+1$ vertices whose maximum degree is strictly smaller than $\sqrt{n}+1$. By Gotsman-Linial's equivalence theorem, this resulted in the existence of boolean functions with $s(f) < \sqrt{\deg(f)}+1$. Interestingly, Huang later proved that $s(f) \geq \sqrt{\deg(f)}$ for every boolean function f.

In the trilean case, we have proved that there are arbitrary large values of n such that there exist three induced subgraphs H_1, H_ε and H_{ε^2} of $\Delta^{\square n}$ with the following properties:

- their vertices constitute a tripartition of $\{1, \varepsilon, \varepsilon^2\}^n$,
- the corresponding H_1', H_ε' and H_{ε^2}' do not verify that $|V(H_1')| = |V(H_\varepsilon')| = |V(H_{\varepsilon^2}')| = 3^{n-1}$, and
- $\Gamma(H_1, H_\varepsilon, H_{\varepsilon^2}) = 2n - \min\{\delta(H_1), \delta(H_\varepsilon), \delta(H_{\varepsilon^2})\} < 2\sqrt{n}+1$.

This construction together with Theorem 6 yields the following:

Proposition 7. *For n sufficiently large there exist trilean functions $f : \{1, \varepsilon, \varepsilon^2\}^n \to \{1, \varepsilon, \varepsilon^2\}$ such that*

$$s(f) < \sqrt{2\deg(f)} + 1.$$

5 Future work

The main open problem now is to determine whether the sensitivity conjecture is true or not in the trilean case, which is essentially a combinatorial problem in graph theory thanks to the equivalence theorem. Furthermore, it would be interesting to generalize all of our results to the case of functions on finite sets of cardinality greater than three.

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