Categorification of Flag Algebras*

Aldo Kiem[1,2], Sebastian Pokutta[1,2], and Christoph Spiegel[1,2]

[1]Technische Universität Berlin, Institute of Mathematics
[2]Zuse Institute Berlin, Department AIS2T, *lastname*@zib.de

Abstract

Razborov introduced the notion of flag algebras in 2007. Since then, they have become an important computational and theoretical tool in extremal combinatorics. Originally phrased in terms of universally quantified first-order theories and often presented purely combinatorially when applied, we propose a category theoretic foundation for flag algebras that unifies these previous approaches. This allows us to obtainin some new foundational results for flag algebras, such as a partial classification of linear and order-preserving maps between them and higher-order differential methods.

1 Introduction

Razborov introduced flag algebras as a unifying language to connect a distinct set of connected problems in extremal combinatorics in 2007 [13]. This was phrased in terms of universally quantified first-order theories and, roughly speaking, is applicable whenever any subset of a structure induces a substructure. Flag algebras also have a strong connection to combinatorial limit objects [4]. Razborov also introduced some fundamental techniques, such as ways to relate different algebras in the form of a downward- and upward uperator, the differential method, a Cauchy-Schwarz-type inequality and the prerequisites for the sum-of-squares method.

While purely theoretical applications of flag algebras have been important, such as the application of the differential method to resolve the minimal density of triangles in graphs [14], the computational approach stemming from the sum-of-squares method has had the most significant impact, covering results relating to 3-edge-colored triangles [5], 4-edge-colored triangles [7], 3-edge-colored rainbow triangles [3], pentagons in triangle-free graphs [6], as well as Turán problems [15], [1], [12] and[10]. These applications often present their own purely combinatorial derivation of flag algebras and their relevant properties. However, some of these ad-hoc derivations, while valid from a combinatorial point of view, are not covered by Razborov's original first-order framework; in particular problems relating to hypercubes [2], [1] and finite vector spaces [11] do not fulfill the property that every subset induces a substructure.

Partially to address this issue, we present a category theoretic foundation for flag algebras. This also answers Coregliano and Razborov's [4] inquiry for a more in-depth study of the category `Int` whose objects are the universally quantified first-order theories with all total interpretations as arrows. It also ties into previous efforts of obtaining a categorical generalisation of combinatorial phenomena as in part the study of graph limits [8] (see chapter 23.4 and the discussion concerning categories and flag algebras). The category theoretic view has the added benefit of resulting in an overall cleaner presentation. Finally, this framework also allows us to obtain some new results regarding the theory

*The full version of this work is currently under preparation. This work was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689).

of flag algebras. In particular, we obtain a partial classification of linear and order-preserving maps between flag algebras, complementing previous efforts in Razborov's original paper, and we formulate higher-order vertex differential methods.

We start by giving the category theoretic formulation of flag algebras and its relation to `Int` in Section 2. We establish their basic properties within this framework in Section 3 and cover some novel statements in Section 4. We conclude with a short discussion in Section 5.

2 Construction

We let `FinInj` denote the category that has all finite sets as its objects and injections between these sets as arrows. Our first observation is that the category `Int` is a full subcategory of the category of presheaves on `FinInj`.

Observation 1. *Given a category $\mathfrak{A}$, denote by* $\texttt{FinPsh}(\mathfrak{A})$ *the category of finite presheaves on* $\mathfrak{A}$*. Let us show how to embedd* `Int` *as a full subcategory of* `FinSet`*. For an object T of* `Int`*, define the finite presheaf* $F_T : \texttt{FinInj}^{\mathrm{op}} \to \texttt{FinSet}$ *where* $F_T(S) = \{M \text{ is a } T\text{-model with ground set } S\}$*. For an arrow* $\alpha : R \to S$ *in* `FinInj`*, we let* $F_T(\alpha)(M)$ *be the unique model with ground set R so that α is an embedding* $F_T(\alpha)(M) \to M$*. The association* $T \to F_T$ *is then a full functor* $\texttt{Int} \to \texttt{FinPsh}(\texttt{FinSet})$*.*

In other words, every total interpretation corresponds to a natural transformation of presheaves and vice versa. In order to get a category theoretic definition of flag algebras we need the technical definition of an Archimedean partially-ordered vector space.

Definition 2. *An $\mathbb{R}$-vector space V with a preordering $\leq$ is* Archimedean *when for every $v, w \in V$ we have that $v \leq r\, w$ for every $r \in \mathbb{R}_{>0}$ implies $v \leq 0$. A linear map $f : V \to W$ between two Archimedean vector spaces is* order-preserving *if $v \leq v'$ in V implies that $f(v) \leq f(v')$ in W. Let* `Arch` *denote the category with all (possibly infinite) Archimedean preordered $\mathbb{R}$-vector spaces as its objects and all order-preserving linear maps as its arrows.*

Definition 3. *The* powering functor $\mathbb{R}^{\cdot} : \texttt{Set} \to \texttt{Arch}$ *is given by mapping any object S of* `Set` *to the Archimedean space $\mathbb{R}^S$ and any arrow $f : R \to S$ of* `Set` *to the map $\mathbb{R}^S \to \mathbb{R}^R$ that sends a function $g : S \to \mathbb{R}$ to $g \circ f$.*

Momentarily disregarding their multiplicative structure, the flag algebras as defined by Razborov in [13, 4] can be seen as a functor $\mathcal{A} : \texttt{Int} \to \texttt{Arch}$. The colimit $\mathcal{A}[F] = \operatorname{colim} \mathbb{R}^F$, which can be shown to exist for any $F \in \texttt{FinPsh}(\mathfrak{A})$ for any category $\mathfrak{A}$, lifts this to a functor $\texttt{FinPsh}(\mathfrak{A}) \to \texttt{Arch}$. In particular, for every universally quantified first-order theory T, the flag algebra $\mathcal{A}[T]$ as defined by Razborov is isomorphic as an Archimedean vector space to $\mathcal{A}[F_T]$ as just defined. The space $\lim \mathbb{R}[F_T]$ corresponds to $\mathbb{R}$-linear combinations of positive homomorphisms.

It remains to establish under what conditions we can define a multiplicative structure on $\mathcal{A}[F]$ in such a way that it has the same properties as the originally defined flag algebras. We note that we choose to put all structural requirements on the base category $\mathfrak{A}$ and none on the nature of the presheaf F, so that we can define an algebra for any $F \in \texttt{FinPsh}(\mathfrak{A})$ assuming the right conditions on $\mathfrak{A}$.

Definition 4. *For a given category $\mathfrak{A}$, we write $x \leq y$, whenever there exists a morphism from x to y. We say that $\mathfrak{A}$ is a* density category *if:*

1. *For any two objects x, y of $\mathfrak{A}$, there are finitely many arrows in $\mathfrak{A}(x, y)$;*
2. *For any two arrows $\alpha, \beta \in \mathfrak{A}(x, y)$, there exists an isomorphism $\gamma \in \mathfrak{A}(y, y)$ such that $\gamma \circ \alpha = \beta$;*
3. *There exists an increasing countable sequence of objects $x_1, x_2, \ldots$ so that for any object x there exists some index i s.t. $x \leq x_i$;*

4. *For any two objects* $y, z \in \mathfrak{A}$ *there exist* $r \geq 0$ *so that for every* $\epsilon > 0$ *there exists* $x_0 \in \mathfrak{A}$ *so that for all* $x_0 \leq x$ *the fraction* $|\mathfrak{A}(y,x)|/|\mathfrak{A}(z,x)|$ *is well-defined and does not differ from* r *by more than* ϵ*.*
5. *For any finite tuple of objects* P *there exists another finite tuple of objects* $\mathrm{co}(P)$ *together with arrows* $\alpha_{x,y} : x \to y$ *for any* $x \in P$ *and* $y \in \mathrm{co}(P)$*, which, for any* $z \in \mathrm{Obj}(\mathfrak{A})$*, induce an isomorphism of sets*
$$\prod_{x \in P} \mathfrak{A}(x,z) \simeq \coprod_{y \in \mathrm{co}\, P} \mathfrak{A}(y,z).$$

We use the term 'density categories', because these are precisely the properties that are needed in order to make sense of the usual notion of densities between structures as well as products of densities. When $\mathfrak{A}$ is a density category, $F \in \mathtt{FinPsh}(\mathfrak{A})$ as well as $f \in F(x)$ and $g \in F(y)$, we can define an equivalent notion to Razborov's homomorphism density through the quotient

$$p(f;g) = |\{\alpha \in \mathfrak{A}(x,y) | F(\alpha)(g) = f\}/|\mathfrak{A}(x,y)|.$$

When $f \in \mathbb{R}^F(x)$ and $g \in \mathbb{R}F(y)$ are linear combinations, we can extend this density bilinearly to $p(f;g)$. We now define a multiplication in $\mathcal{A}[F]$.

Definition 5. *Let* $F \in \mathtt{FinPsh}(\mathfrak{A})$*. Given a tuple* $f_1 \in F(x_1), \dots, f_m \in F(x_m)$ *and* $P = (x_1, \dots, x_m)$*, define their product as*
$$f_1 \cdots f_m = \sum_{y \in \mathrm{co}(P)} g_y \lim_z \frac{|\mathfrak{A}(y,z)|}{\prod_{x \in P} |\mathfrak{A}(x,z)|} \in \mathcal{A}[F]$$
where $g_y \in \mathbb{R}^{F(y)}$ *is the unweighted sum of all* $g \in F(y)$ *so that for* $i = 1, \dots m$*, we have* $f_i = g \circ F(a_{x_i,y})$*.*

We let $1 \in \mathcal{A}[F]$ denote the sum of all flags corresponding to an arbitrary but fixed source object y. Note that the definition of $1 \in \mathcal{A}[F]$ is independent of the choice of the source object y and multiplication in $\mathcal{A}[F]$ is associative, commutative, and has 1 as its unit. Furthermore, squares are positive in $\mathcal{A}[F]$.

Examples for density categories include the aforementioned `FinSet` as well as finite vector spaces $\mathtt{FinVec}_q$ over some finite field $\mathbb{F}_q$ with injective arrows and the category `HyperCube` of hypercubes with morphisms the injective face maps. In Observation 1 we showed how to convert a combinatorially meaningful object T in `Int` to an object of `FinPsh(FinInj)`. The same theme can be used to convert previously studied theories over $\mathtt{FinVec}_q$ and `HyperCube` to finite presheaves. For example, c-vertex colorings of the objects of $\mathtt{FinVec}_q$ are represented by that $F \in \mathtt{FinPsh}(\mathtt{FinVec}_q)$ which maps a vector space V to the set $F(V)$ of all c-vertex colorings of V, not up to isomorphism. Similarly, c-edge colorings of the objects of `HyperCube` are represented by that $F \in \mathtt{FinPsh}(\mathtt{HyperCube})$ mapping a hypercube C to the set $F(C)$ of all c-edge colorings of the edges of C, not up to isomorphism.

3 Properties

All of the flag algebra calculus remains true when we are considering a finite presheaf $F \in \mathtt{FinPsh}(\mathfrak{A})$ over a density category. In particular, there are downward operators. Since squares are always positive, this means that the sum of squares method works for any such F.

Definition 6. *Let* $P : \mathfrak{A} \to \mathfrak{B}$ *be a functor between density categories. The* downward operator *between two flag algebras* $[\![\cdot]\!]_P : \mathcal{A}[F \circ P] \to \mathcal{A}[F]$ *is defined as the map induced by sending any* $f \in (F \circ P)(x)$ *to the same element in* $F(P(x))$*.*

The notion of types for flag algebras is essential and the basis for the sum-of-squares method. What they allow for is an amalgamated multiplication of flag aglebra elements. This produces elements in the positive cone of $\mathcal{A}[F]$ that would be extremely difficult to detect otherwise. In the context of the categorification of flag algberas, types are given by coslice categories. Let $\mathfrak{A}$ be a density category and $x \in \mathrm{Obj}(\mathfrak{A})$ so that the under category $x/\mathfrak{A}$ is again a density category. Then, denote by U_x the forgetful functor $x/\mathfrak{A} \to \mathfrak{A}$. The algebra at x is $\mathcal{A}^x[F] = \mathcal{A}[F \circ U_x]$ and the downward operator between the algebras $\mathcal{A}^x[F] \to \mathcal{A}[F]$ is given by Definition 6. Similarly, when $\alpha : x \to y$ is a morphism, there is an upward operator $\pi^\eta : \mathcal{A}^x[F] \to \mathcal{A}^y[F]$. It is defined by multiplying an element $f \in \mathcal{A}^x[F]$ with a projection of the unit of $\mathcal{A}^y[F]$ and interpreting the result as an element of $\mathcal{A}^y[F]$ again. The upward and downward operators are compatible in the same ways as in [13].

Problems in extremal combinatorics usually speak of a certain limit of combinatorial structures that minimizes or maximizes an objective function. In terms of Razborov's flag algebras, these limit sequences are represented by order-preserving algebra homomorphisms $\mathcal{A}[F_T] \to \mathbb{R}$ and vice versa. Several of the properties we require of a density category have the sole purpose of ensuring that this correspondence remains true for $F \in \mathtt{FinPsh}(\mathfrak{A})$.

Definition 7. *For an increasing sequence $x_1, x_2, \ldots$ of objects in $\mathfrak{A}$ as in Item 3 and elements $u_n \in \mathbb{R}F(x_n)$, we say that u_n is convergent if for every flag f of $\mathcal{A}[F]$ the limit of $p(f; u_n)$ as $n \to \infty$ exists.*

Then, by our definition of density categories we get the following equivalent statement to Razborov's Theorem 3.3 from [13]. For two partially ordered algebras A and B, let $\mathrm{Hom}^+(A, B)$ denote the set of order-preserving maps from A to B.

Theorem 8. *Let $x_1, x_2, \ldots$ be a countable sequence of objects in $\mathfrak{A}$ as in Item 3. There is a surjective correspondence between $\mathrm{Hom}^+(\mathcal{A}[F], \mathbb{R})$ and convergent sequences $u_n \in F(x_n)$.*

4 Results

In this section, we investigate what order-preserving, not necessarily multiplicative, morphisms $\mathcal{A}[F] \to \mathcal{A}[G]$ there exist when $F, G \in \mathtt{FinPsh}(\mathfrak{A})$. Some of these maps are already known, like the downward operators or the the image under $\mathcal{A}$ of a natural transformation $G \to F$ of presheaves. We will focus on showing how to classify the particular case of natural transformations $\mathbb{R}^F \to \mathbb{R}^G$. Interestingly, when $\mathfrak{A} = \mathtt{FinInj}$, the presheaf G is G_T for some theory T like edge colored u-uniform hypergraphs without forbidden substructures, these natural transformations $\mathbb{R}^F \to \mathbb{R}^G$ coincide bijectively with the elements of $\mathrm{Hom}^+(\mathcal{A}_{[G,F]}, \mathbb{R})$ where $[G, F]$ is the internal hom of presheaves. In general however, $\mathrm{Hom}^+(\mathcal{A}_{[G,F]}, \mathbb{R})$ does not classify the natural transformations $\mathbb{R}^F \to \mathbb{R}^G$ for arbitrary presheaves F and G, even over $\mathfrak{A} = \mathtt{FinInj}$.

First we show that there are many ways to represent every natural transformation $\mathbb{R}^F \to \mathbb{R}^G$ injectively as a convergent sequence in the presheaf $\mathbb{R}[F \times G]$. Intuitively, we think of $\mathbb{R}[F \times G]$ as the space in which we represent the graph of a natural transformation $\mathbb{R}^F \to \mathbb{R}^G$, just as it would be when we consider the graph in $A \times B$ of a function $A \to B$ when A and B are sets.

Remark 9. *For a natural transformation $L : \mathbb{R}^F \to \mathbb{R}^G$, denote by $L^\vee$ the dual natural transformation $\mathbb{R}[G] \to \mathbb{R}[F]$. Let $\psi \in \mathrm{Hom}^+(\mathcal{A}[G], \mathbb{R})$ be a positive homomorphism that is non-zero on every element of G and let $u_i \in G(x_i)$ be a sequence that converges to ψ.*

Consider the convergent sequence in $\mathbb{R}[F \times G]$

$$w_i = L^\vee(u_i) \times u_i.$$

From such a sequence w_i, we can uniquely recover the values $L^\vee(g)$ for $g \in G(x)$ through the formula

$$L^\vee(g) = \tfrac{1}{\psi(g)} \sum_{f \in F(x)} \left(\lim_i p(f \times g; w_i) \right) f. \tag{1}$$

This is a consequence of the fact that L is a natural transformation. Therefore, given ψ, every natural transformation $L : \mathbb{R}^F \to \mathbb{R}^G$ corresponds to a unique convergent sequence in $\mathbb{R}[F \times G]$.

The problem is that given $\psi \in \mathrm{Hom}^+(\mathcal{A}[G], \mathbb{R})$, we do not know which convergent sequences in $\mathbb{R}[F \times G]$ correspond to natural transformations L. Therefore, we introduce the technical notion of a vertex extension property for ψ which guarantees a converse.

Theorem 10. *Assume that $\psi \in \mathrm{Hom}^+(\mathcal{A}[G], \mathbb{R})$ has the vertex extension property for all elements of G and let $u_i \in G(x_i)$ be a sequence that converges to ψ. Then for every sequence $v_i \in \mathbb{R}[F(x_i)]$ so that $v_i \times u_i$ converges in $\mathbb{R}[F \times G]$, Equation* (1) *defines a natural transformation.*

The precise definition of the vertex extension property is given in the following. Example homomorphisms that fulfill this condition are the u-uniform random hypergraphs $\psi \in \mathrm{Hom}^+(\mathcal{A}[F_{u\text{-Hyper}}], \mathbb{R})$. Therefore, we get a complete description of all natural transformations $\mathbb{R}^F \to \mathbb{R}^{F_{u\text{-Hyper}}}$ when $F \in$ `FinPsh(FinInj)`.

Definition 11. *Assume that $\mathfrak{A}$ and $x/\mathfrak{A}$ are density categories and let $f \in F(x)$. Let $\psi \in \mathrm{Hom}^+(\mathcal{A}[F], \mathbb{R})$ with $\psi(f) > 0$. Recall that we denote the forgetful functor $x/\mathfrak{A} \to \mathfrak{A}$ by U_x. For any y and $\eta : x \to y$ as well as $g \in F(y)$ denote by $\langle g \rangle_\eta \in (F \circ U_x)(\eta)$ the same object g but viewed as an element in a presheaf over a coslice category.*

We say that ψ has the vertex extension property for f if there exists a sequence $u_i \in F(x_i)$ converging to ψ so that for every $\alpha : x \to y$ and $h \in F \circ U_x(\alpha)$ with $F(\alpha)(\llbracket h \rrbracket_{U_x}) = f$, the sequence of random variables $E_i \circ \beta_i$

$$E_i \circ \beta_i = p(h; \langle u_i \rangle_{\beta_i}) - \frac{\psi(\llbracket h \rrbracket_{U_x})}{\psi(f)} p(f; F(\beta_i)(u_i))$$

with uniformly independent $\beta_i \in \mathfrak{A}(x, x_i)$, converges almost surely to 0 as i tends to infinity.

Finally, we study a special case of linear and order-preserving maps that give rise to the vertex differential method. When $\mathfrak{A} =$ `FinInj` and the presheaf is F_T, the map arises from the notions we have just developed as follows. Denote by M_n the multiplication endofunctor `FinInj` $\to$ `FinInj` that maps $S \mapsto S \times [n]$. The vertex differential method then arises from a composition

$$\mathcal{A}[F_T] \to \mathcal{A}[F_T \circ \mathrm{M}_n] \to \mathcal{A}[F_T]$$

where the first arrow is induced by a natural transformation $\mathbb{R}^{F_T} \to \mathbb{R}^{F_T \circ \mathrm{M}_n}$ and the second is the downward operator. We will however simply write out the defining formula of this linear and order-preserving map directly.

Lemma 12. *We work over $\mathfrak{A} =$ `FinInj` and denote by $\mathbf{1} \in$ `FinInj` the set $\{1\}$. Let T be a universally quantified first-order theory. Choose any $h \in \mathcal{A}^{\mathbf{1}}[F_T]$ with $h \geq -1$ and $\llbracket h \rrbracket_{U_{\mathbf{1}}} = 0$. For $f \in F_T(y)$ define*

$$V(f) = \left\llbracket \langle f \rangle_{\mathrm{id}_y} \prod_{\alpha \in \mathfrak{A}(\mathbf{1}, y)} \left(1 + \pi^\alpha(h)\right) \right\rrbracket_{U_y}.$$

Then, V is linear order-preserving and its derivatives as $h \to 0$ are the (higher order) differential methods.

The fact that V is order-preserving follows from $h \geq -1$. The first-order method was already discovered by Razborov in [13]. All higher-order methods are novel. It is also possible to get higher-order edge-differential methods. To this end we must consider the functors $\mathrm{B}_u :$ `FinInj` $\to$ `FinInj` which map $S \mapsto \binom{S}{u}$.

5 Discussion

We have shown that the theory of flag algebras very naturally carries over to the setting of density categories. However, several interesting avenues of exploration remain. It would for example be interesting to explore, if one can fully classify the homomorphism $\mathrm{Hom}^+(\mathcal{A}[F], \mathbb{R})$ as Razborov and Coregliano [4] did for canonical universally quantified theiories. Furthermore, Razborov [13] was able to show that so called *open* interpretations give a class of maps between certain localizations of flag algebras. It would be interesting to see how our observations regarding the classification of natural transformations $\mathbb{R}^F \to \mathbb{R}^G$ would carry over to the case of localizations. In fact, one can use the endofunctor M_n to construct some basic maps into localizations, but we have not pursued this line of thought any further at this point.

The fact that we have chosen our presheaves to take values in finite sets is essential for the basic features of Flag Algebras. We are grateful to an anonymous referee for directing our attention to the structural limits of [9]. Stated briefly in our language, given a countable signature λ, define a measure space valued presheaf F on `FinInj`

$$F(S) = (\{(\mathbf{A}, v) \mid \mathbf{A} \text{ is a finite } \lambda\text{-structure on } S \text{ and } v : [n] \to S\}, D, \mu_{\text{count}})$$

where D is the σ-algebra generated by all subsets of $F(S)$ that are definable by $\mathrm{FO}[\lambda]$ formulas with free variables in $\{x_s \mid s \in S\}$. The measure μ_{count} is the counting measure that gives weight 1 to each equivalence class of the $(\mathbf{A}, v)$. Then, the basic objects of study of [9] are $\lim L^1(F)$, corresponding to the finite λ-structures, the flag algebra $\mathcal{A}[F]$ on $\operatorname{colim} L^\infty(F)$ corresponding to the Lindenbaum-Tarski algebra and its dual $\lim \mathrm{ba}(F)$.

References

[1] Rahil Baber. Turán densities of hypercubes. *arXiv preprint arXiv:1201.3587*, page 161171, 2012.

[2] József Balogh, Ping Hu, Bernard Lidický, and Hong Liu. Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube. *European Journal of Combinatorics*, 35:75–85, 2014. Selected Papers of EuroComb'11.

[3] József Balogh, Ping Hu, Bernard Lidický, Florian Pfender, Jan Volec, and Michael Young. Rainbow triangles in three-colored graphs. *Journal of Combinatorial Theory, Series B*, 126:83–113, 2017.

[4] Leonardo Nagami Coregliano and Alexander Razborov. Semantic limits of dense combinatorial objects. *Russian Mathematical Surveys*, 75(4):627, 2020.

[5] James Cummings, Daniel Král', Florian Pfender, Konrad Sperfeld, Andrew Treglown, and Michael Young. Monochromatic triangles in three-coloured graphs. *Journal of Combinatorial Theory, Series B*, 103(4):489–503, 2013.

[6] Hamed Hatami, Jan Hladkỳ, Daniel Král', Serguei Norine, and Alexander Razborov. On the number of pentagons in triangle-free graphs. *Journal of Combinatorial Theory, Series A*, 120(3):722–732, 2013.

[7] Aldo Kiem, Sebastian Pokutta, and Christoph Spiegel. The four-color ramsey multiplicity of triangles. *arXiv preprint arXiv:2312.08049*, 2023.

[8] László Lovász. *Large networks and graph limits*, volume 60. American Mathematical Soc., 2012.

[9] Jaroslav Nešetřil and Patrice Ossona de Mendez. *A unified approach to structural limits and limits of graphs with bounded tree-depth*, volume 263. American Mathematical Society, 2020.

[10] Olaf Parczyk, Sebastian Pokutta, Christoph Spiegel, and Tibor Szabó. New Ramsey multiplicity bounds and search heuristics. *arXiv preprint arXiv:2206.04036*, 2022.

[11] Juanjo Rue Perna and Christoph Spiegel. The rado multiplicity problem in vector spaces over finite fields. In *Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications, EUROCOMB'23*, pages 784 – 789, 2023.

[12] Oleg Pikhurko and Emil R Vaughan. Minimum number of k-cliques in graphs with bounded independence number. *Combinatorics, Probability and Computing*, 22(6):910–934, 2013.

[13] Alexander Razborov. Flag algebras. *The Journal of Symbolic Logic*, 72(4):1239–1282, 2007.

[14] Alexander Razborov. On the minimal density of triangles in graphs. *Combinatorics, Probability and Computing*, 17(4):603–618, 2008.

[15] Alexander Razborov. On 3-hypergraphs with forbidden 4-vertex configurations. *SIAM Journal on Discrete Mathematics*, 24(3):946–963, 2010.