Bicolored point sets admitting non-crossing alternating Hamiltonian paths*

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Abstract

Consider a bicolored point set P in general position in the plane consisting of red and blue points such that the number of blue points differs from the number of red points by at most one. We show that if a subset of the red points forms the vertices of a convex polygon separating the blue points, lying inside the polygon, from the remaining red points, lying outside the polygon, then the points of P can be connected by non-crossing straight-line segments so that the resulting graph is a properly colored Hamiltonian path.

1 Introduction

In geometric graph theory it is a common problem to decide whether a given graph can be drawn in the plane on a given point set so that the edges are represented by non-crossing straight-line segments. For example, deciding whether a given general planar graph has a non-crossing straight-line drawing on a given point set is NP-complete [7].

There are many interesting unanswered questions when considering bicolored point sets instead (see the comprehensive survey by Kano and Urrutia [11]). We restrict ourselves to drawings of bipartite graphs on bicolored point sets where edges are drawn as non-crossing straight-line segments between points of different colors. This question remains interesting even for paths. Let B and R denote a set of blue points and a set of red points in the plane, respectively, such that $R \cup B$ is in general position, i.e., no three points are collinear. We call a non-intersecting path on $R \cup B$ whose edges are straight-line segments and every segment connects two points of $R \cup B$ of distinct colors, an *alternating path.* If such an alternating path connects all points of $R \cup B$, we call it an *alternating Hamiltonian path.* If such an alternating Hamiltonian path shares the first and last vertex (but otherwise is still non-intersecting), we call it an *alternating Hamiltonian cycle.*

If $||R| - |B|| \leq 1$ and R can be separated from B by a line, then Abellanas et al. [1] showed that there always exists an alternating Hamiltonian path on $R \cup B$. This fact together with the well-known Ham sandwich theorem implies that if $|R| = |B|$, then there always exists an alternating path on $R \cup B$ connecting at least half of the points. This trivial lower bound on the length of an alternating path that always exists is the best known according to our knowledge. This bound was improved by a small linear factor by Mulzer and Valtr [12] for point sets in *convex* position, i.e., when the points form the vertices of a convex polygon. On the other hand, if we do not assume that R and B are separated by a line, then there are examples where $|R| = |B| \geq 8$ and no alternating Hamiltonian path on $R \cup B$ exists, even if $R \cup B$ is in convex position. Moreover, for $R \cup B$ in convex position with $|R| = |B| = n$, Csóka et al. [9] showed that there are configurations where the longest alternating path on $R \cup B$ has size at most $(4 - 2\sqrt{2})n + o(n)$.

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As we have seen above, an alternating Hamiltonian path does not exist on every point set but it exists if $||R| - |B|| \leq 1$ and R can be separated from B by a line. Another sufficient condition was found by Cibulka et al. [8]. They looked more closely at configurations where R and B form a so-called double chain and showed that if $||R| - |B|| \leq 1$ and each chain of the double-chain contains at least one-fifth of all points, then there exists an alternating Hamiltonian path on $R \cup B$.

The final sufficient condition that we know of was found by Abellanas et al. [1]. They showed that if $||R| - |B|| \leq 1$, the points of R are vertices of a convex polygon, and all points of B are inside this polygon, then there exists an alternating Hamiltonian path on $R \cup B$.

In this paper, we generalize this last result, and by doing so, we extend the known family of configurations of points for which there exists an alternating Hamiltonian path on $R \cup B$. Specifically, we prove the following theorem.

Theorem 1. *Let R be a set of red points and B be a set of blue points such that $R \cup B$ is in general position. Let P be a convex polygon whose vertices are formed by a subset of R. Assume that the remaining points of R lie outside of P, points of B lie in the interior of P, and $||R| - |B|| \leq 1$. Then there exists an alternating Hamiltonian path on $R \cup B$.*

When $|R| = |B|$ we even find an alternating Hamiltonian cycle.

2 Preliminaries and an outline of the proof

By a *polygonal region* we understand a closed, possibly unbounded, region in the plane whose boundary (possibly empty) consists of finitely many non-crossing straight-line segments or half-lines connected into a polygonal chain. A bounded polygonal region is a polygon. A polygon can be defined by an ordered set of its vertices; in that case, we assume that the vertices lie on the boundary of the polygon in the clockwise direction, and we use index arithmetic modulo the number of vertices. A *diagonal* of a convex polygon (or a polygonal region) is any segment connecting two points on the boundary of the polygon. For an edge e of a convex polygon (or polygonal region), the closed half-plane to the side of e that is disjoint with the polygon's interior is denoted by $\mathrm{out}(e)$. For two points a, b in the plane, we denote by ab the segment connecting them. The *convex hull* of a set of points X, denoted by $\mathrm{conv}(X)$, is the smallest convex set that contains X.

Recall that B and R always denote the set of blue points and the set of red points, respectively. Moreover, B and R are always disjoint, and $R \cup B$ is always in general position. For a region T of the plane, $\|T\|_R$ and $\|T\|_B$ denotes the number of red points inside T and the number of blue points inside T, respectively. For the points on the boundaries of regions, we specify if they belong to the region or not (we will need to assign every point to exactly one part of a partition of the plane into polygonal regions).

Our primary result, Theorem 1, is a generalization of the following theorem proved by Abellanas et al. [1].

Theorem 2 ([1])**.** *Let R be a set of red points and B be a set of blue points such that $R \cup B$ is in general position. Let R form the vertices of the polygon $\mathrm{conv}(R \cup B)$, the points of B lie in the interior of $\mathrm{conv}(R \cup B)$, and $||R| - |B|| \leq 1$. Then there exists an alternating Hamiltonian path on $R \cup B$.*

Our improvement lies in the fact that the polygon P can be formed by a subset of R (instead of the whole R), whereas the remaining points of R remain outside of P. The approach in the proof of Theorem 2 in the case when $|R| = |B|$ is to split the polygon formed by R into convex polygons, each containing exactly one edge of the polygon and one blue point from inside the polygon, and then connect by straight-line segments each of the blue points to the vertices of the edge that is inside the same part. In this way, alternating paths of length two are formed inside each part of the partition. Moreover, they share their end vertices, and so, together, they form an alternating Hamiltonian cycle (this cycle is non-crossing since each of the small paths lies in its own part of the partition).

We proceed similarly with only one significant distinction. Namely, we partition the whole plane into convex parts so that every edge of the polygon is a diagonal of one part, and each part contains one more blue point than it contains red points (not counting the vertices of the polygon). Inside each of these parts, we find an alternating Hamiltonian path. And these paths together form an alternating Hamiltonian cycle as before.

In section 3 we outline how we split the plane and in section 4 how to find the alternating Hamiltonian path.

3 Partitioning theorem

For the partitioning of the plane, we prove the following theorem.

Theorem 3. *Let $P = (p_1, \dots, p_s)$ be a convex polygon, B be a set of blue points in the interior of P, and R be a set of red points outside of P such that $s = |B| - |R|$ and $R \cup B \cup \{p_1, \dots p_s\}$ is in general position. Then there exists a partition of the plane into convex polygonal regions $Q_1, \dots, Q_s$ such that each $p_i p_{i+1}$ is a diagonal of Q_i and for every i, we have $\|Q_i\|_B - \|Q_i\|_R = 1$. Moreover, every point of $R \cup B$ is counted in exactly one Q_i. That is, if a point of $R \cup B$ lies on the common boundary of more Q_i's it is assigned to only one of them.*

For the case of $s = 3$, i.e., when P is a triangle, we prove the following stronger lemma.

Lemma 4. *Let Q be a convex polygonal region, $P = (p_1, p_2, p_3)$ be a triangle inside Q, B be a set of blue points in the interior of P, and R be a set of red points outside P but inside Q such that $R \cup B \cup \{p_1, p_2, p_3\}$ is in general position. Additionally let $n_1, \dots, n_3$ be integers satisfying the following conditions.*

1. *$|B| - |R| = n_1 + n_2 + n_3$.*
2. *For every nonempty subset I of $\{1, \dots, 3\}$,*

$$\sum_{i \in I} n_i \geq - \left\| Q \cap \bigcup_{i \in I} \operatorname{out}(p_i p_{i+1}) \right\|_R . \quad (1)$$

Then there exists a point y in P different from p_1, p_2 and p_3 such that the half-lines yp_1, yp_2 and yp_3 split Q into three parts Q_1, Q_2 and Q_3. Moreover, there exists an assignment of points of $B \cup R$ that lie on the boundaries of Q_1, Q_2 and Q_3 into adjacent parts so that $\|Q_i\|_B - \|Q_i\|_R = n_i$.

For an example partition according to Lemma 4, see Figure 1.

Note that the conditions established by Inequation (1) are necessary: When I contains only one index i, the part Q_i is split by $p_i p_{i+1}$ into two regions, one inside P and one outside of P. The part outside of P is inside $\operatorname{out}(p_i p_{i+1})$, and so $\|Q_i\|_R \leq \|Q \cap \operatorname{out}(p_i p_{i+1})\|_R$. Together with a trivial condition $\|Q_i\|_B \geq 0$ we get $\|Q_i\|_B - \|Q_i\|_R \geq - \|Q \cap \operatorname{out}(p_i p_{i+1})\|_R$, which is exactly one of the conditions. It can be analogously observed for larger cardinalities of I's.

Furthermore, note that in the case when Q is the plane and all n_i's are equal to 1, the conditions established by Inequation (1) always hold, and Lemma 4 implies Theorem 3 when P is a triangle.

In the proof of Lemma 4, we employ a standard technique (see Akiyama and Alon [2]) and substitute points with disks of the same area and work with the area of the disks instead of the number of points. This is helpful because the boundaries of polygonal regions have an area of size zero, and so the area of all disks will be precisely distributed between the interiors of the polygonal regions of the partition. We find the point y using a well-known result in fixed point theory: Knaster–Kuratowski–Mazurkiewicz lemma (see [6, Theorem 5.1] for a simple proof). At the end of the proof, we return from disks back to points and we have to solve the problem where to assign points whose disks intersect the boundaries of the partition. We present the details in the full version.

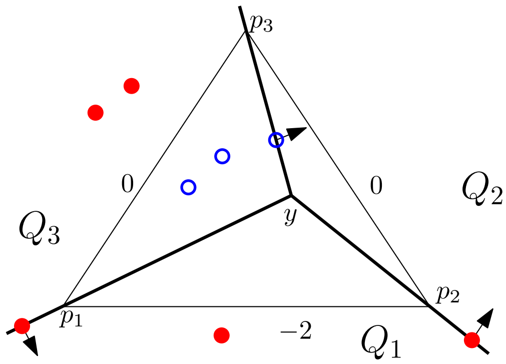

Figure 1: Illustration of a partition from Lemma 4. Region Q_1 contains two fewer blue points than it contains red points. Region Q_2 contains the same number of blue points as red points. And the same holds for Q_3. The arrows indicate to which regions belong the points on the boundaries.

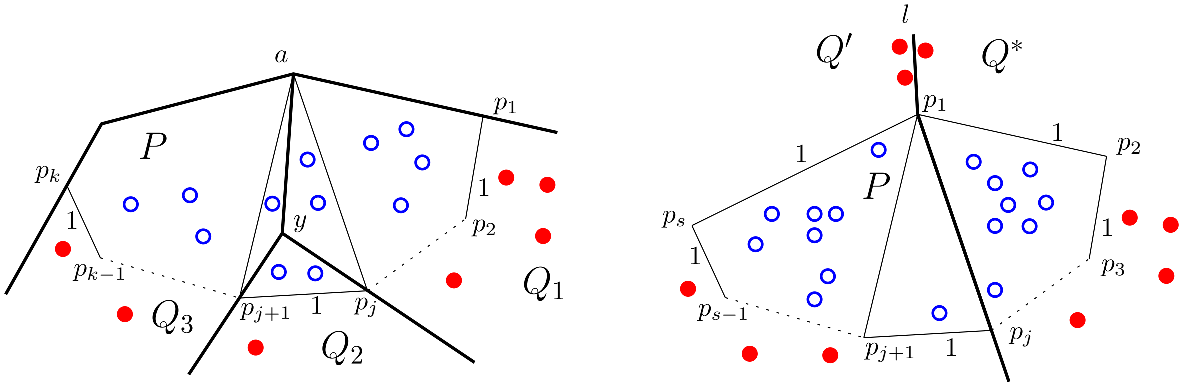

Figure 2: Left: Inside the polygon P we find a triangle ap_jp_{j+1} and apply Lemma 4 to split the polygonal region into three parts Q_1, Q_2, Q_3. The induction hypothesis can then be applied to Q_1 and Q_3 to obtain a complete partition of the polygonal region.
Right: In the first step of the partitioning of the convex polygon P we sometimes use a half-line l and partition regions Q^* and Q' by induction.

To prove Theorem 3, we use induction on the number of vertices of the polygon P. The main idea is to find a suitable triangle formed by vertices of P, and apply Lemma 4 to this triangle. We set the numbers n_1, n_2, n_3 so that we obtain three polygonal regions Q_1, Q_2 and Q_3 each containing $\|Q_i\|_B - \|Q_i\|_R$ edges of P. Then we partition Q_1, Q_2 and Q_3 by induction. Note that except for the very first step, we are partitioning bounded polygonal regions instead of the plane, and the polygon P is already partially split but that is only easier. See Figure 2, left.

Unfortunately, finding the very first triangle is not always possible. However, if that is the case, then we can find a diagonal p_1p_j of P and a half-line l shooting from p_1 such that l and the half-line p_1p_j split the plane into two polygonal regions Q^* and Q' that can be partitioned by the normal induction process. See Figure 2, right. We present the details in the full version.

4 Conclusion of the proof

To finish the proof of Theorem 1, we use a result proved by Abellanas et al. [1] about point sets with color classes separated by a line. We use a slightly modified version that easily follows from the proof of the original version.

Theorem 5 ([1])**.** *Let R be a set of red points and B be a set of blue points such that $R \cup B$ is in*

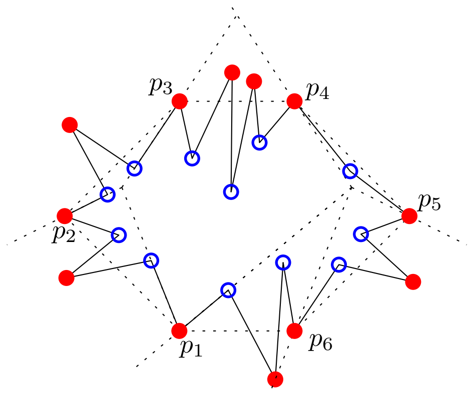

Figure 3: An alternating Hamiltonian cycle in a case when 6 red points form a polygon separating the remaining 6 red points from 12 blue points lying inside the polygon.

general position. Assume that $|R| - |B| = 1$ and that there are two points $r_1, r_2 \in R$ such that the line r_1r_2 separates R from B and that r_1, r_2 are vertices of the convex hull $\operatorname{conv}(R \cup B)$. Then there exists an alternating Hamiltonian path on $R \cup B$ with end vertices r_1, r_2.

We apply this theorem several times to the partition obtained by Theorem 3 to finish the proof of Theorem 1.

(Idea of the) proof of Theorem 1. We may assume that $|R| = |B|$ and prove that there exists an alternating Hamiltonian cycle, otherwise, we could add one point and remove it at the end. Let $R' = R \setminus P$. That is, R' contains exactly the points of R that are not vertices of P. Therefore, $s = |B| - |R'|$.

By Theorem 3 applied on the polygon P, the set of blue points B and the set of red points R', there exists a partition of the plane into convex polygonal regions $Q_1, \ldots, Q_s$ such that for every i, the edge p_ip_{i+1} is a diagonal of Q_i, and for every i, the region Q_i contains exactly one more blue point than red points of R'.

By Theorem 5 applied to each Q_i separately, we obtain an alternating Hamiltonian path in each Q_i with ends in p_i and p_{i+1} covering all red and blue points inside Q_i. These paths are connected together in the end vertices p_i. Therefore, together they form an alternating Hamiltonian cycle. See Figure 3 for an illustration.

□

5 Conclusion and open questions

The main technical part of our proof is Theorem 3. We believe that the following stronger version that also generalizes Lemma 4 holds.

Conjecture 6. *Let Q be a convex polygonal region, $P = (p_1, \ldots, p_s)$ be a convex polygon inside Q, B be a set of blue points in the interior of P, and R be a set of red points outside P but inside Q such that $R \cup B \cup \{p_1, \ldots p_s\}$ is in general position. Additionally let $n_1, \ldots, n_s$ be integers satisfying the following conditions.*

1. *$|B| - |R| = n_1 + \cdots + n_s$.*
2. *For every nonempty cyclic interval of indices I from $\{1, \ldots, s\}$,*

$$\sum_{i \in I} n_i \geq - \left\| Q \cap \bigcup_{i \in I} \operatorname{out}(p_ip_{i+1}) \right\|_R . \qquad (2)$$

Then there exists a partition of Q into convex polygonal regions $Q_1, \ldots, Q_s$ such that for every i, the segment $p_i p_{i+1}$ is a diagonal of Q_i and $\|Q_i\|_B - \|Q_i\|_R = n_i$. Moreover, every point of $R \cup B$ is counted in exactly one Q_i.

Similarly, as for Lemma 4 we observe that the conditions established by Inequation (2) are necessary.

The case when there are no red points outside of P and every n_i is a positive integer was already proved by García and Tejel [10] and later by Aurenhammer [3]. The case with points outside of P seems to be more difficult (for example, even some negative n_i's can satisfy the conditions established by Inequation (2) in that case).

Similar problems of finding partitions of colored point sets into subsets with disjoint convex hulls such that the sets of points of all color classes are partitioned as evenly as possible is well studied, see [4, 5]. However, we were not able to apply the results directly because we have the additional restriction that $p_i p_{i+1}$'s have to be diagonals of the convex hulls in the partition. We managed to prove Conjecture 6 only in the case when $s = 3$ (Lemma 4) and that proved crucial in proving Theorem 3.

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