The rectilinear convex hull of disks

Carlos Alegría [1], Justin Dallant [2], Jean-Paul Doignon [2], Pablo Pérez-Lantero [3], and Carlos Seara [4]

[1]Dip. Ingegneria, Università Roma Tre, carlos.alegria@uniroma3.it
[2]Dept. Computer Science, Université libre de Bruxelles, {justin.dallant,paul.doignon}@ulb.be
[3]Depto. Matemática y Computación, Universidad de Santiago de Chile, pablo.perez.l@usach.cl.
[4]Dept. Matemàtiques, Universitat Politècnica de Catalunya, Spain, carlos.seara@upc.edu

Abstract

We explore an extension to orthogonal convexity of the classic problem of computing the convex hull of a collection of planar disks. Namely, we enumerate all the changes to the boundary of the rectilinear convex hull of a collection of n planar disks, while the coordinate axes are simultaneously rotated by an angle that goes from 0 to 2π. Our algorithm takes $\Theta(n \log n)$ time and $\Theta(n)$ space.

1 Introduction

Let D denote a collection of n closed planar disks. The convex hull of D, which we denote by $\mathcal{CH}(D)$, is the region obtained by removing from the plane all the open half-planes whose intersection with D is empty. The *rectilinear convex hull* of D, which we denote by $\mathcal{RCH}(D)$, is instead the region obtained by removing from the plane all the axis-aligned open wedges of aperture angle $\frac{\pi}{2}$, whose intersection with D is empty (a formal definition is given in Section 2). See Figure 1.

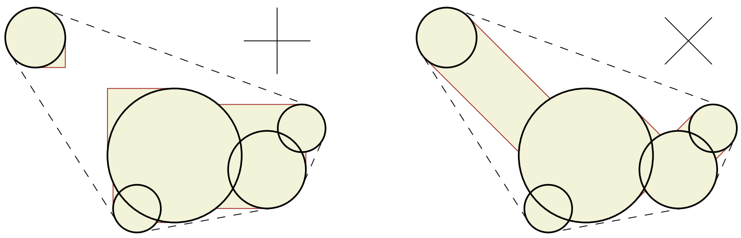

Figure 1: A set D of closed disks and $\mathcal{RCH}(D)$ for two orientations of the coordinate axes. The axes are shown in the top-right corner of each figure. The edges of $\mathcal{CH}(D)$ are shown with dashed lines. The interior and the edges of $\mathcal{RCH}(D)$ are shown respectively, in light and dark brown. On the left, $\mathcal{RCH}(D)$ has two connected components. On the right, $\mathcal{RCH}(D)$ has a single connected component.

The rectilinear convex hull is the analog of the (standard) convex hull on a non-traditional notion of convexity called *orthogonal convexity* [6]. In this notion of convexity, convex sets are restricted to those whose intersection with any horizontal or vertical line is either empty, a point, or a line segment. The rectilinear convex hull introduces two important differences with respect to the standard convex hull. On one hand, note that $\mathcal{RCH}(D)$ may be a simply connected set, yielding an intuitive and appealing structure. However, if the union of D is disconnected, then $\mathcal{RCH}(D)$ may have several simply connected components. On the other hand, observe that the orientation of the empty wedges changes along with the orientation of the coordinate axes, changing the shape of $\mathcal{RCH}(D)$ as well. The former property has shown to be useful to better separate finite sets of points that are not separable by a line or even by a standard convex hull [1]. The later property has been used to explore a family of problems in which the rotation angle of the coordinate axes is used as a search space for optimization criteria [2].

Let $\mathcal{RCH}_\theta(D)$ denote the rectilinear convex hull of D computed after simultaneously rotating the coordinate axes by an angle θ (a formal definition is given in Section 2). Let $\partial(S)$ denote the boundary of a planar set S. In this paper we describe an $\Theta(n \log n)$-time and $\Theta(n)$-space algorithm that enumerates the changes to $\partial(\mathcal{RCH}_\theta(D))$ while θ is increased from 0 to 2π. Notably, despite the introduction of rotations to the coordinate axes, our algorithm successfully achieves the complexities of the well known algorithm to compute the standard convex hull of a collection of planar disks [5].

2 Preliminaries

The *orientation* of a line is the smallest of the two possible angles it makes with the X^+ positive semiaxis. A *set of orientations* is a set of lines with different orientations passing through some fixed point. Hereafter we consider a set of orientations formed by two orthogonal lines. For the sake of simplicity, we assume that both lines are passing through the origin and are parallel to the coordinate axes. We denote such an orientation set with $\mathcal{O}$. We say that a planar region is *$\mathcal{O}$-convex*, if its intersection with a line parallel to a line of $\mathcal{O}$ is either empty, a point, or a line segment.

Let ρ_1 and ρ_2 be two rays with a common apex point $x \in \mathbb{R}^2$ such that, after rotating ρ_1 around x by an angle of $\omega \in [0, 2\pi)$, we obtain ρ_2. We refer to the two open regions in the set $\mathbb{R}^2 \setminus (\rho_1 \cup \rho_2)$ as *wedges*. We say that both wedges have vertex x and sizes ω and $2\pi - \omega$, respectively. A *quadrant* is a wedge of size $\frac{\pi}{2}$ whose rays are parallel to the lines of $\mathcal{O}$. We say that a planar region is *free of points of D*, or *D-free* for short, if it contains no point of a disk of D. Let $\mathcal{O}_\theta$ denote the set resulting after simultaneously rotating the lines of $\mathcal{O}$ in the counterclockwise direction by an angle of θ. Let $\mathcal{Q}_\theta$ be the set of all (open) D-free quadrants of the plane whose rays are parallel to the lines of $\mathcal{O}_\theta$.

Definition 1. *The rectilinear convex hull of D* with respect to $\mathcal{O}_\theta$, *is the closed and $\mathcal{O}_\theta$-convex set*

$$\mathcal{RCH}_\theta(D) = \mathbb{R}^2 \setminus \bigcup_{q \in \mathcal{Q}_\theta} q.$$

We assume that $\mathcal{O}_0 = \mathcal{O}$ and $\mathcal{RCH}_0(D) = \mathcal{RCH}(D)$.

As mentioned in Section 1, there are two main differences between the standard and the rectilinear convex hull. First, for any fixed value of θ we have that $\mathcal{RCH}_\theta(D)$ may be non-convex and may even be formed by several connected components; see again Figure 1. Each component is closed, simply connected, $\mathcal{O}_\theta$-convex, and is either a disk (if D is a singleton) or a region bounded by a *curvilinear polygon*, which is a simple polygon whose edges are either line segments or circular arcs. If an edge is a line segment, then it belongs to the boundary of a D-free quadrant. If it is instead a circular arc, then it belongs to the boundary of a disk of D. The second difference is a property we call *orientation dependency*: except for particular values of an angle α, such as multiples of $\frac{\pi}{2}$, we have that $\mathcal{RCH}_\theta(D) \neq \mathcal{RCH}_{\theta+\alpha}(D)$, $\alpha \in [0, 2\pi)$.

From an algorithmic point of view, $\mathcal{CH}(D)$ is described by a circular list of (possibly repeated) disks of D, sorted by appearance as we traverse $\partial(\mathcal{CH}(D))$ in counterclockwise direction. From this list we can trivially obtain $\partial(\mathcal{CH}(D))$ in linear time; see [5] for more details. We describe $\partial(\mathcal{RCH}_\theta(D))$ for fixed values of θ in a similar way. Instead of a single list, we use four disjoint lists containing each a set of circular arcs of the disks of D, sorted by appearance as we traverse $\partial(\mathcal{RCH}_\theta(D))$. To formally describe these lists, we use a simple (yet crucial) observation that derives from Definition 1. An ω-wedge is a wedge of size at least ω. We say a point $x \in \mathbb{R}^2$ is *ω-wedge D-free*, if there exists a D-free ω-wedge with apex at x.

Observation 2. *Consider a fixed value of θ. A point x of (a disk of) D lies on the boundary of $\mathcal{RCH}_\theta(D)$ if, and only if, it is $\frac{\pi}{2}$-wedge D-free.*

The *N-orientation* (for North-orientation) through a point $x \in \mathbb{R}^2$ is the ray pointing upwards starting at x. The *S-orientation*, *E-orientation*, and *W-orientation* through a point are defined analogously. A point $x \in \mathbb{R}^2$ is *ω-wedge D-free* with respect to the N-orientation, if there is a D-free ω-wedge with apex at x that contains the N-orientation through x. The same definition can be analogously given for the remaining three orientations.

When computing $\partial(\mathcal{RCH}_\theta(D))$, there are two types of changes as θ is increased from 0 to 2π: *combinatorial changes*, where there is a change on the ordered list of circular arcs of D on $\partial(\mathcal{RCH}_\theta(D))$; and *geometric changes*, where there is no change on the list, but only on the coordinates of the endpoints of circular arcs. Geometric changes between two combinatorial changes can be accounted for by representing the circular arc endpoints as (known) functions of θ, instead of fixed values. We thus focus on computing combinatorial changes.

3 Rectilinear convex hull of a set of disks

Let $\mathcal{N}(D)$ denote the *upper envelope* of D, that is, the set of points of D seen from the north infinity. Analogously, let $\mathcal{S}(D)$, $\mathcal{E}(D)$, and $\mathcal{W}(D)$ denote respectively, the envelopes of D seen from the South, East, and West infinities. To compute and maintain $\partial(\mathcal{RCH}_\theta(D))$ we proceed as follows. We first compute $\mathcal{N}(D)$, $\mathcal{S}(D)$, $\mathcal{E}(D)$, and $\mathcal{W}(D)$. From the four envelopes we compute the points of D that are $\frac{\pi}{2}$-wedge D-free with respect to the N-, S-, W-, and E- orientations. We combine the information computed into a data structure to compute the four fronts for any value of θ. Then, we traverse this data structure increasing θ from 0 to 2π while computing all the combinatorial changes in $\partial(\mathcal{RCH}_\theta(D))$.

Computing the envelopes. The complexity of each envelope is $O(n)$ since the union of the disks of D has linear complexity [4]. Each envelope can be computed in $O(n \log n)$ time and $O(n)$ space [3].

The set of points that are $\frac{\pi}{2}$-wedge D-free. Consider in the following the envelope $\mathcal{N}(D)$; see Figure 2. The remaining three envelopes are similarly processed. The envelope is formed by a sequence of x-monotone circular arcs sorted by appearance while sweeping the plane from left to right. Two consecutive arcs may share and endpoint, and no vertical line intersects the interior of two arcs.

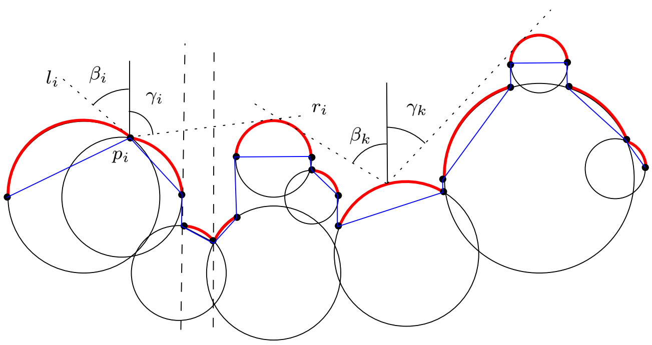

Figure 2: The upper envelope $\mathcal{N}(D)$ of a set of circles.

We first show how to decide which endpoints of the arcs of $\mathcal{N}(D)$ are $\frac{\pi}{2}$-wedge D-free. Let a_i denote the ith arc of $\mathcal{N}(D)$, and p_i, p_{i+1} denote the endpoints of a_i. For each endpoint p_i, we compute the wedge with apex at p_i that contains the N-orientation through p_i, and has the biggest possible size ω_i. We keep the endpoints for which $w_i \geq \frac{\pi}{2}$, as well as the corresponding w_i-wedge. Let $\leq$ denote the weak order on the endpoints induced by the values of their abscissas. We proceed as follows.

1. Sweep $\mathcal{N}(D)$ from left to right doing the following while visiting each endpoint p_i. Let D_i denote the subset of $\mathcal{N}(D)$ whose arcs have endpoints p_j such that (i) $p_j \leq p_i$ if p_i is a right endpoint of an arc, and (ii) $p_j < p_i$ otherwise. Update $\mathcal{CH}(D_i)$ in $O(\log n)$ time, and add p_i to D_i.

Notice that we have to update $\mathcal{CH}(D_i)$ from $\mathcal{CH}(D_{i-1})$ by either: (i) adding an arc $a_i = (p_{i-1}, p_i)$ to $\mathcal{CH}(D_{i-1})$ and computing the corresponding bridge, or (ii) adding a point p_i to $\mathcal{CH}(D_{i-1})$ by computing the supporting line to $\mathcal{CH}(D_{i-1})$ from p_i.

We do this computation in $O(n)$ overall amortized time since we use the order of the arcs in $\mathcal{N}(D)$ and walk along $\mathcal{N}(D)$ till we find the arcs to define the bridge contained in the supporting line. Once we find the arcs, the bridge can be computed in constant time using elementary geometry.

2. Compute the supporting line l_i from p_i to $\mathcal{CH}(D_{i-1})$ by traversing the boundary of $\mathcal{CH}(D_{i-1})$ until we find the tangent vertex of $\mathcal{CH}(D_{i-1})$. The tangent vertex can be either an endpoint in $\mathcal{N}(D)$ or a point in an arc in $\mathcal{N}(C)$, in the last case compute this point and the supporting line in constant time. At the end of the sweep, this process amortizes to $O(n)$ in time and space. We also compute the angle β_i formed by l_i and the line with N-orientation passing through p_i.

3. We can proceed doing the same computation but considering the right to left sorting, and maintaining D'_{i-1} which is defined symmetrically. Then, we compute the supporting line r_i from p_i to $\mathcal{CH}(D'_{i-1})$, and compute the angle γ_i formed by r_i and the line with N-orientation passing through p_i. Again, at the end of the sweep this process amortizes to $O(n)$ in time and space.

4. Compute $\omega_i = \beta_i + \gamma_i$, and check whether $\omega_i \geq \frac{\pi}{2}$. In the affirmative, let ω_i be the *angular interval* associated with p_i. For each p_i such that $\omega_i \geq \frac{\pi}{2}$, we form the angular interval for p_i as the intersection with the unit circle of the image of the wedge by mapping p_i to the origin.

Since there are $O(n)$ endpoints, the complexity of these steps is $O(n)$ time and space. We proceed analogously with the envelopes $\mathcal{S}(D)$, $\mathcal{E}(D)$, and $\mathcal{W}(D)$. Thus, the total complexity for this process for the four envelopes is $O(n)$ time and space. We are considering the at most $O(n)$ endpoints p_i of the four envelopes and computing which of these endpoints p_i have $\omega_i \geq \frac{\pi}{2}$ for some of the four orientations above. For each such point and orientation, we record its angular interval (as it is defined a above in the case of the northern orientation). From the discussion above we have the following result.

Theorem 3. *The $O(n)$ endpoints p_i of the envelopes $\mathcal{N}(D)$, $\mathcal{S}(D)$, $\mathcal{E}(D)$, and $\mathcal{W}(D)$ having an angle $\omega_i \geq \frac{\pi}{2}$, their angles ω_i, and their angular intervals can be computed in $O(n \log n)$ time and $O(n)$ space.*

We next show how to compute the interior points of the arcs of D that belong $\partial(\mathcal{RCH}_\theta(D))$. For each arc $a_i = (p_{i-1}, p_i)$ of $\mathcal{N}(D)$, we will show how to compute the angles β and γ for the interior points of a_i, and therefore, how to compute the parts of a_i (if any) whose interior points have angles β and γ such that $\beta + \gamma = \omega \geq \frac{\pi}{2}$.

Given an arc $a_i = (p_{i-1}, p_i)$ of $\mathcal{N}(D)$, we proceed as in the item 1. For the endpoint p_i of a_i we update in $O(\log n)$ time the $\mathcal{CH}(D_i)$, where D_i is the set of arcs in $\mathcal{N}(D)$ with the endpoints p_j such that (i) $p_j \leq p_i$ if p_i is a right endpoint of an arc, or (ii) $p_j < p_i$ otherwise adding p_i to D_i.

Proceeding with the endpoints of the arc $a_i = (p_{i-1}, p_i)$, assume that we have computed the left support line l_{i-1} and the right support line r_{i-1} from the endpoint p_{i-1}, and analogously, l_i and r_i from the endpoint p_i. Then, because we maintain the $\mathcal{CH}(D_i)$ and $\mathcal{CH}(D'_i)$ for a_i, we can split a_i into sub-arcs such that from the points inside each sub-arc we have a unique left (resp. right) supporting arc in $\mathcal{N}(D)$ for computing the respective supporting lines. In each sub-arc we know the left and right arcs that support the left (resp. right) supporting lines from the endpoints of a sub-arc in a_i. Next, we determine whether and where the points inside these sub-arcs having angle $w \geq \frac{\pi}{2}$. We parameterize the calculus of the angle ω in each sub-arc as a function of the angle θ that it is illustrated in Figure 3.

The blue curve in Figure 3 Right is a Limaçon curve, see Sánchez-Ramos et al. [7], also known as a Limaçon of Pascal or Pascal's Snail. The cardioid is a special case. We have drawn this curve for two circles in Figure 3 Right. The part of the Limaçon curve that we are interesting in is the part of the curve that is in between the two red circles, say c_j and c_k, and defined by the intersection point of the perpendicular lines which are tangents lines to c_j and c_k, as illustrated in Figure 3 Left.

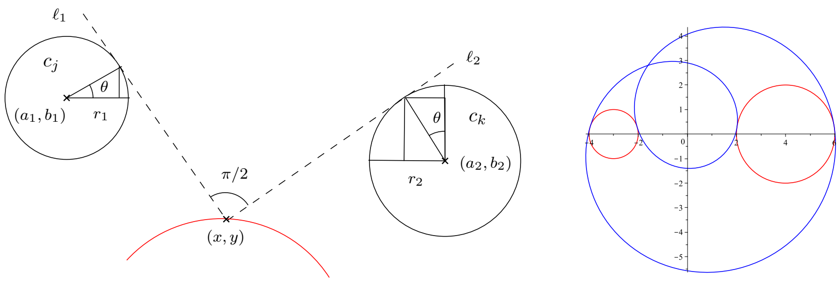

Figure 3: Left: the angle $\omega = \pi/2$ in a point (x, y) of a sub-arc a_i. Right: The Limaçon curve with angle $\pi/2$, in blue. The red circles contain the two arcs of the envelope $\mathcal{N}(D)$. The left circle has center $(-3, 0)$ and radius 1 and the right circle has center $(4, 0)$ and radius 2.

We compute the solutions of the equations and determine the intervals. In constant time, we can check whether the values of ω inside the computed intervals verify that $\omega \geq \frac{\pi}{2}$, and thus, we determine the constant number of intervals where $\omega \geq \frac{\pi}{2}$. We can proceed with the other envelopes $\mathcal{S}(D)$, $\mathcal{E}(D)$, and $\mathcal{W}(D)$. Therefore, the total complexities for all together are $O(n \log n)$ time and $O(n)$ space.

Theorem 4. *The $O(n)$ intervals in the arcs of circles in the envelopes $\mathcal{N}(D)$, $\mathcal{S}(D)$, $\mathcal{E}(D)$, and $\mathcal{W}(D)$ whose interior points have an angle $\omega \geq \frac{\pi}{2}$, their angles ω, and the corresponding angular intervals can be computed in $O(n \log n)$ time and $O(n)$ space.*

The data structure. We translate all the angular intervals for all the arcs of D in $\mathcal{N}(D)$, $\mathcal{S}(D)$, $\mathcal{E}(D)$, and $\mathcal{W}(D)$, to angular intervals inside $[0, 2\pi]$ on the real line. In this way, we can do a line sweep with four vertical lines corresponding to angles θ, $\theta + \frac{\pi}{2}$, $\theta + \pi$, and $\theta + \frac{3\pi}{2}$ in a circular way (i.e., completing a $[0, 2\pi]$ round with each line), and then inserting and deleting the changes of the arcs (or part of them) that belong to $\partial(\mathcal{RCH}_\theta(D))$ as θ changes in $[0, 2\pi)$.

Now, considering that an endpoint or an interior point of an arc in any of the envelopes $\mathcal{N}(D)$, $\mathcal{S}(D)$, $\mathcal{E}(D)$, and $\mathcal{W}(D)$ can be the apex of a D-free $\frac{\pi}{2}$-wedge, from Theorems 3 and 4, we conclude that we can compute the points of disks of D that belong to $\partial(\mathcal{RCH}_\theta(D))$ as $\theta \in [0, 2\pi]$ in $O(n \log n)$ time and $O(n)$ space. From this discussion we obtain the main result of our paper.

Theorem 5. *Given a set D of n closed disks in the plane, computing and maintaining $\partial(\mathcal{RCH}_\theta(D))$ as θ is increased from 0 to 2π can be done in $O(n \log n)$ time and $O(n)$ space.*

References

[1] Alegría, C., Orden, D., Seara, C., Urrutia, J.: Separating bichromatic point sets in the plane by restricted orientation convex hulls. J. Global Optim. **85**, 1003–1036 (2023)

[2] Alegría-Galicia, C., Orden, D., Seara, C., Urrutia, J.: Efficient computation of minimum-area rectilinear convex hull under rotation and generalizations. J. Global Optim. **70**(3), 687–714 (2021)

[3] Devillers, O., Golin, M.J.: Incremental algorithms for finding the convex hulls of circles and the lower envelopes of parabolas. Information Processing Letters **56**(3), 157–164 (1995)

[4] Kedem, K., Livne, R., Pach, J., Sharir, M.: On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles. Discrete Comput. Geom. **1**(1), 59–71 (1986)

[5] Rappaport, D.: A convex hull algorithm for discs, and applications. Computational Geometry **1**(3), 171–187 (1992)

[6] Rawlins, G.J., Wood, D.: Ortho-convexity and its generalizations. In: Computational Morphology, Machine Intelligence and Pattern Recognition, vol. 6, pp. 137–152. North-Holland (1988)

[7] Sánchez Ramos, I., Meseguer-Garrido, F., Aliaga, J.J., Grau, J.: Generalization of the pedal concept in bidimensional spaces. Application to the limaçon of Pascal. DYNA **88**, 196–202 (2021)