A note on generalized crowns in linear r-graphs*

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Abstract

An r-graph H is called linear if any two edges of H intersect in at most one vertex. Let F and H be two linear r-graphs. If H contains no copy of F, then H is called F-free. The linear Turán number of F, denoted by $ex_r^{lin}(n,F)$, is the maximum number of edges in any F-free n-vertex linear r-graph. The crown $C_{1,3}$ is a linear 3-graph which is obtained from three pairwise disjoint edges by adding one edge that intersects all three of them in one vertex. In 2022, Gyárfás, Ruszinkó and Sárközy initiated the study of $ex_3^{lin}(n,F)$ for different choices of an acyclic 3-graph F. They established lower and upper bounds for $ex_3^{lin}(n,C_{1,3})$. In this paper, we generalize the notion of a crown to linear r-graphs for $r \geq 3$, and also generalize the above results to linear r-graphs.

1 Introduction

The result presented here is motivated by a number of very recent papers on linear Turán numbers. We extend a result on crown-free linear 3-graphs to linear r-graphs for $r \geq 3$. Throughout, we let r be an integer with $r \geq 3$.

Let $H = (V,E)$ be an r-graph consisting of a set of vertices $V = V(H)$ and a collection $E = E(H)$ of r-element subsets of V called edges. If any two edges in H intersect in at most one vertex, then H is said to be linear. Let F be a linear r-graph. Then H is called F-free if it contains no copy of F as its subhypergraph. The linear Turán number of F, denoted by $ex_r^{lin}(n,F)$, is the maximum number of edges in any F-free linear r-graph on n vertices. More generally, for two linear r-graphs F_1 and F_2, H is called $\{F_1,F_2\}$-free if it contains no copy of F_1 or F_2 as its subhypergraph. The linear Turán number of $\{F_1,F_2\}$, denoted by $ex_r^{lin}(n,\{F_1,F_2\})$, is the maximum number of edges in any $\{F_1,F_2\}$-free linear r-graph on n vertices.

A linear 3-graph is acyclic if it can be constructed in the following way. We start with one edge. Then at each step we add a new edge intersecting the union of the vertices of the previous edges in at most one vertex. In 2022, Gyárfás, Ruszinkó and Sárközy [5] initiated the study of $ex_3^{lin}(n,F)$ for different choices of an acyclic 3-graph F. In [5], they determined the linear Turán numbers of linear 3-graphs with at most 4 edges, except the crown, for which they gave lower and upper bounds (Theorem 1 below). Here the crown is a linear 3-graph which is obtained from three pairwise disjoint edges on 3

*The full version of this work can be found in [10] and will be published elsewhere. This research is supported by the National Natural Science Foundation of China (No. 12271439) and China Scholarship Council (No. 202206290003).

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vertices by adding one edge that intersects all three of them in one vertex. In [5], the authors used E_4 to denote a crown, but here we adopt the notation $C_{1,3}$ from the more recent paper [9].

Since the publication of [5], there have appeared several results involving the linear Turán number of some acyclic linear hypergraphs [6, 7, 8]. In the remainder, we focus on results involving $C_{1,3}$, as our aim is to present a natural generalization of these results to linear r-graphs.

In [5], Gyárfás, Ruszinkó and Sárközy obtained the following result.

Theorem 1 ([5]).

$$6\left\lfloor \frac{n-3}{4} \right\rfloor + \varepsilon \leq ex_3^{lin}(n, C_{1,3}) \leq 2n,$$

where $\varepsilon = 0$ *if* $n-3 \equiv 0, 1 \pmod 4$*,* $\varepsilon = 1$ *if* $n-3 \equiv 2 \pmod 4$*, and* $\varepsilon = 3$ *if* $n-3 \equiv 3 \pmod 4$*.*

Indeed, for the lower bound in Theorem 1, the authors of [5] gave the following construction for obtaining a class of extremal linear $C_{1,3}$-free 3-graphs. We recall this construction for later reference. Start with the graph mK_4 consisting of m disjoint copies of the complete graph on four vertices. The graph mK_4 admits a one-factorization, *i.e.*, a decomposition of the edge set into three edge-disjoint perfect matchings. Each of these matchings corresponds to $2m$ vertex-disjoint pairs of edges. Add one new vertex for each of the matchings and form $2m$ triples by adding this vertex to each of the $2m$ pairs. Now ignore the edges of the mK_4. This construction consists of $n = 4m + 3$ vertices and $6m$ triples, and it is easy to check that the corresponding 3-graph is linear and $C_{1,3}$-free. Thus for $n = 4m+3$, this construction provides an extremal 3-graph with $6\left\lfloor \frac{n-3}{4} \right\rfloor + \varepsilon$ edges, where ε is defined as in the above theorem. The construction can be adjusted to obtain extremal 3-graphs for the other residue classes modulo 4.

In a later paper [2], Carbonero, Fletcher, Guo, Gyárfás, Wang, and Yan proved that every linear 3-graph with minimum degree 4 contains a crown. The same group of authors conjectured in [1] that $ex_3^{lin}(n, C_{1,3}) \sim \frac{3n}{2}$, and proposed some ideas to obtain the exact bounds. After that, Fletcher [4] improved the upper bound to $ex_3^{lin}(n, C_{1,3}) \leq \frac{5n}{3}$.

Very recently, Tang, Wu, Zhang and Zheng [9] established the following result.

Theorem 2 ([9]). *Let G be any C_{13}-free linear 3-graph on n vertices. Then $|E(G)| \leq \frac{3(n-s)}{2}$, where s denotes the number of vertices in G with degree at least 6.*

The above result shows that the lower bound in Theorem 1 is essentially tight. Furthermore, the above result, combined with the results in [5], essentially completes the determination of the linear Turán numbers for all linear 3-graphs with at most 4 edges.

2 Crown-free linear r-graphs

In the remainder, we focus on the following natural generalization of the notion of a crown to linear r-graphs. An r-crown $C_{1,r}$ is a linear r-graph on r^2 vertices and $r+1$ edges obtained from r pairwise disjoint edges on r vertices by adding one edge that intersects all of them in one vertex. In fact, for our purposes we need a second generalization of the crown to linear r-graphs. We let $C_{1,r}^*$ denote the following linear r-graph on $r^2 - r + 3$ vertices and $r+1$ edges. It consists of a set of $r-2$ edges $\{e_1, e_2, \ldots, e_{r-2}\}$ that intersect in exactly one vertex v, two additional disjoint edges e_{r-1} and e_r that are also disjoint from $\{e_1, e_2, \ldots, e_{r-2}\}$, and one additional edge e intersecting each edge of $\{e_1, e_2, \ldots, e_r\}$ in exactly one vertex except for v. Note that both $C_{1,r}$ and $C_{1,r}^*$ are isomorphic to the crown in case $r = 3$.

In the following, we establish an upper bound on $ex_r^{lin}(n, \{C_{1,r}, C_{1,r}^*\})$, and a lower bound on $ex_r^{lin}(n, \{C_{1,r}, C_{1,r}^*\})$ when $r-1$ is a prime power.

In order to obtain a lower bound on $ex_r^{lin}(n, \{C_{1,r}, C_{1,r}^*\})$, we can use a similar construction as in the description following Theorem 1. We can construct a $\{C_{1,r}, C_{1,r}^*\}$-free linear r-graph on n vertices by using the notion of a transversal design.

Assume that n is a multiple of k for some integer $k \geq r-1$. A transversal design $T(n,k)$ is a linear k-graph on n vertices, in which the vertices are partitioned into k sets, each containing $\frac{n}{k}$ vertices, and where each pair of vertices from different sets belongs to exactly one edge on k vertices. Note that $T(n,k)$ is an $\frac{n}{k}$-regular k-partite linear k-graph. It can be found in [3] that such $T(n,k)$ exist for sufficiently large n when k divides n. In particular, $T(k^2,k)$ exists when k is a prime power.

Let $r-1$ be a prime power. Denote by $T'((r-1)^2, r-1)$ the linear $(r-1)$-graph obtained from $T((r-1)^2, r-1)$ by adding one edge for each set in the partition. Note that for $r=3$, $T'((r-1)^2, r-1)$ is a K_4. We next extend m disjoint copies of $T'((r-1)^2, r-1)$ to a $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph in the same way as we did for $r=3$ starting with mK_4. Consider a one-factorization of the linear $(r-1)$-graph $mT'((r-1)^2, r-1)$. Each of the r factors corresponds to $(r-1)m$ vertex-disjoint $(r-1)$-tuples. Add one new vertex for each of the factors and form $(r-1)m$ edges by adding this vertex to each of the $(r-1)m$ $(r-1)$-tuples. The resulting linear r-graph has $r(r-1)m$ edges and $(r-1)^2m+r$ vertices, and it is $\{C_{1,r}, C^*_{1,r}\}$-free. Let $n=(r-1)^2m+r$. Then the number of edges of the constructed r-graph is at least $r(r-1)\left\lfloor \frac{n-r}{(r-1)^2} \right\rfloor$, where $r-1$ is a prime power.

In order to obtain an upper bound on $ex_r^{lin}(n, \{C_{1,r}, C^*_{1,r}\})$, we generalize the result of Theorem 2 to linear r-graphs. We present our proof of the following theorem in the next section. In the final section, we complete the paper with a short discussion.

Theorem 3. *Let G be any $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph on n vertices, and let s denote the number of vertices with degree at least $(r-1)^2+2$. Then $|E(G)| \leq \frac{r(r-2)(n-s)}{r-1}$.*

3 Proof of Theorem 3

For the full proof see manuscript [10].

Before we present our proof, we need some additional notation, and we prove a key lemma. Let H be a linear r-graph, let $d_1 \geq d_2 \geq \ldots \geq d_r$ be positive integers, and let $e \in E(H)$. Then we use $D(e) \geq \{d_1, d_2, \ldots, d_r\}$ to denote that e can be written as $e=\{u_1, u_2, \ldots, u_r\}$ such that $d(u_i) \geq d_i$ for each $i \in [r] = \{1, 2, \ldots, r\}$. Here $d(v)$ denotes the degree, i.e., the number of edges containing the vertex v. We use the shorthand v-edge for an edge containing the vertex v.

Lemma 4. *Let G be a $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph, and let $e \in E(G)$ be such that $D(e) \geq \{(r-1)^2+1, (r-1)^2+1, (r-1)^2, \ldots, (r-1)^2\}$. Then*

$$S = \bigcup_{f \in E(G), f \cap e \neq \emptyset} f$$

contains exactly $(r-1)^3+r$ vertices, and all vertices in S have degree at most $(r-1)^2+1$. Moreover,

$$E_S = \{f : f \in E(G), f \cap S \neq \emptyset\}$$

contains at most $r(r-1)^2+1$ edges.

Proof. Without loss of generality, suppose $e=\{u_1, u_2, \ldots, u_r\}$ with $d(u_1) \geq d(u_2) \geq (r-1)^2+1$ and $d(u_i) \geq (r-1)^2$ for each $3 \leq i \leq r$. If $d(u_1) \geq (r-1)^2+2$, we can find a copy of $C_{1,r}$ in the following way. We start with the edge $e=\{u_1, u_2, \ldots, u_r\}$. We can find a u_r-edge $e_1 \neq e$ since $d(u_r) \geq (r-1)^2$. By considering i from $r-1$ to 2 one by one, we can find a u_i-edge e_{r-i+1} that does not share a vertex with any edge in $\{e_1, e_2, \ldots, e_{r-i}\}$. Finally, we can choose a u_1-edge e_r that does not share a vertex with $e_1, e_2, \ldots, e_{r-1}$. Hence, we have found a copy of $C_{1,r}$, a contradiction.

Therefore, we have $d(u_1)=d(u_2)=(r-1)^2+1$. For $p \in \{u_1, u_2, \ldots, u_r\}$, we use $G(p)$ to denote the set of all vertices outside e that lie on a common edge with p. Firstly, we have the following claim. (Due to page limitations, we omit the proofs for the following claims.)

Claim 3.1. $G(u_1) = G(u_2)$.

Similarly, we must have $G(u_i) \subset G(u_2)$ for each $3 \le i \le r$. Suppose to the contrary that there exists some $3 \le i \le r$ such that there is a u_i-edge $e_i \ne e$ containing some vertex not in $G(u_2)$. Then there are at most $r-2$ u_2-edges other than e intersecting e_i, so there are at least $(r-2)(r-1)+1$ u_2-edges that are disjoint from e_i. By the edge conditions that $d(u_1) \ge (r-1)^2+1$ and $d(u_s) \ge (r-1)^2$ for each $3 \le s \le r$, for each s satisfying the conditions $1 \le s \le r, s \ne 2$ and $s \ne i$ we can choose a u_s-edge e_s that is disjoint from $\{e_1, e_3, \ldots, e_{s-1}\}$, and then choose a u_2-edge e_2 that is disjoint from $\{e_1, e_3, \ldots, e_r\}$. So $\{e, e_1, e_2, \ldots, e_r\}$ forms a $C_{1,r}$, a contradiction.

Thus $S \setminus \{u_1, u_2, \ldots, u_r\} = G(u_2) = G(u_1) \supset G(u_i)$ for each $3 \le i \le r$. Denote by F the edge set each edge of which is disjoint from $\{u_1, u_2, \ldots, u_r\}$ and contains at least one vertex of S. It suffices to show that F must be empty.

For this purpose, we first construct $r-1$ auxiliary bipartite graphs as follows. Fix an h with $2 \le h \le r$, and let $H_h = (V_{H_h} = X_{H_h} \cup Y_{H_h}, E_{H_h})$, where $X_{H_h} = \{e_i | u_h \in e_i, e_i \ne e\}$, $Y_{H_h} = \{e_j | u_1 \in e_j, e_j \ne e\}$ and $E_{H_h} = \{\{e_i, e_j\} | e_i \cap e_j \ne \emptyset\}$. Then H_2 is an $(r-1)$-regular bipartite graph with partition classes of exactly $(r-1)^2$ vertices. For $3 \le h \le r$, H_h is a bipartite graph with one class of exactly $(r-1)^2$ vertices and the other class having at least $(r-1)^2 - 1$ vertices. Next, we prove two claims on the structure of these bipartite graphs.

Claim 3.2. *If G is $C_{1,r}$-free, then H_2 must contain a $K_{r-1,r-1}$.*

Claim 3.3. *If G is $C_{1,r}$-free, then H_h must contain a $K_{r-2,r-1}$ for each $2 \le h \le r$. Furthermore, the partition classes on $r-1$ vertices in these $K_{r-2,r-1}$'s are mutually disjoint.*

Let $\{e_1, e_2, \ldots, e_{(r-1)^2}\}$ denote the ordered sequence of all u_1-edges except for e. Without loss of generality, we assume that H_h contains the $(h-1)$-th $r-1$ u_1-edges of this sequence for $2 \le h \le r$. That means H_h contains $e_{(h-2)(r-1)+1}, e_{(h-2)(r-1)+2}, \ldots, e_{(h-1)(r-1)}$ for each $2 \le h \le r$. Denote by U_{h-1} the set of vertices in the $(h-1)$-th $r-1$ u_1-edges of the sequence for $2 \le h \le r$. We have another claim.

Claim 3.4. *Fix $2 \le i \le r$. Each u_i-edge contains only vertices of one vertex set from $\{U_1, U_2, \ldots, U_{r-1}\}$.*

Before we continue with the proof of Lemma 4, we note that the above analysis implies the following about the structure of H_i.

Remarks 3.1. *H_2 is the disjoint union of $r-1$ complete bipartite graphs $K_{r-1,r-1}$. Since $d(u_h) \ge (r-1)^2$ for each $3 \le h \le r$, H_h is either the disjoint union of $r-1$ complete bipartite graphs $K_{r-1,r-1}$ or the disjoint union of $r-2$ complete bipartite graphs $K_{r-1,r-1}$ and one complete bipartite graph $K_{r-2,r-1}$.*

As a consequence of Remarks 3.1, for each $1 \le i \le r-1$ there exist $r-1$ u_2-edges whose vertices except for u_2 are in U_i. Fix h with $3 \le h \le r$. There exists at most one s with $1 \le s \le r-1$ such that there exist $r-2$ u_h-edges whose vertices except for u_h are in U_s. For each $1 \le i \ne s \le r-1$, there exist $r-1$ u_h-edges whose vertices except for u_h are in U_i.

Now we are ready to prove the statement about F. If F is not an empty set, we let f be an edge of F. There must exist an s with $1 \le s \le r-1$ such that $|f \cap U_s| \ge 1$. Let $v \in f \cap U_s$. We choose a u_1-edge g containing v. By Remarks 3.1, there exist $r-2$ u_t-edges $g_1, g_2, \ldots, g_{r-2}$ with the property that each of them is disjoint from f and each of them intersects g. And there must exist another u_1-edge g' whose vertices except for u_1 are in U_t for some $1 \le t \ne s \le r-1$ such that g' is disjoint from f. Now the edges $f, g, g', g_1, g_2, \ldots, g_{r-2}$ constitute a $C^*_{1,r}$, a contradiction. This completes the proof of Lemma 4. □

Now we are ready to prove Theorem 3. Suppose to the contrary that G is a smallest (in terms of the number of vertices n) $\{C_{1,r}, C^*_{1,r}\}$-free linear r-graph such that G has more than $\frac{r(r-2)(n-s)}{r-1}$ edges. For each $v \in V(G)$, we define $I(v) = 1$ if $d(v) \le (r-1)^2 + 1$, and $I(v) = 0$ otherwise.

We adopt the following useful observation from [9].

$$\sum_{e\in E(G)}\sum_{v\in V(G),v\in e}\frac{I(v)}{d(v)}=\sum_{v\in V(G)}\sum_{e\in E(G),v\in e}\frac{I(v)}{d(v)}=\sum_{v\in V(G)}I(v)=n-s.$$

Since $|E(G)|>\frac{r(r-2)(n-s)}{r-1}$, there must exist an edge $e=\{u_1,u_2,\dots,u_r\}$ such that

$$\sum_{1\le i\le r}\frac{I(u_i)}{d(u_i)}<\frac{r-1}{r(r-2)}=\frac{r-1}{(r-1)^2-1}. \tag{1}$$

Without loss of generality, we assume $d(u_1)\ge d(u_2)\ge\dots\ge d(u_r)$. Note that $d(u_r)\ge r-1$ and $d(u_2)\ge(r-1)^2$, as otherwise (1) would be violated. We can also deduce that $d(u_i)\ge(r-i)(r-1)+2$ for all $3\le i\le r-1$, as otherwise (1) would be violated. If $d(u_1)\ge(r-1)^2+2$, then we can easily find a $C_{1,r}$ in the following way. We start with the edge $e=(u_1,u_2,\dots,u_r)$. We can find a u_r-edge $e_1\ne e$ since $d(u_r)\ge 2$. By considering i from $r-1$ to 2 one by one, we can find a u_i-edge e_{r-i+1} that does not share a vertex with any edge in $\{e_1,e_2,\dots,e_{r-i}\}$. Finally, we can choose a u_1-edge e_r that does not share a vertex with $\{e_1,e_2,\dots,e_{r-1}\}$, a contradiction. Therefore, we have $d(u_1)\le(r-1)^2+1$. By (1), we have $d(u_1)=d(u_2)=(r-1)^2+1$ and $d(u_i)\ge(r-1)^2$ for each $3\le i\le r$. Thus, $D(e)\ge\{(r-1)^2+1,(r-1)^2+1,(r-1)^2,\dots,(r-1)^2\}$.

Now we define S and E_S as in Lemma 4. Let $G-S$ be the linear r-graph obtained by deleting the vertices of S and the edges of E_S. By Lemma 4, $G-S$ has $n'=n-((r-1)^3+r)$ vertices and at least $|E(G)|-(r(r-1)^2+1)$ edges. Furthermore, the number of vertices in $G-S$ of degree at least $(r-1)^2+2$ is exactly s. Therefore, we have

$$|E(G-S)|\ge|E(G)|-(r(r-1)^2+1)>\frac{r(r-2)(n-s)}{r-1}-(r(r-1)^2+1)>\frac{r(r-2)(n'-s)}{r-1},$$

which contradicts the assumption that G is a smallest counterexample to Theorem 3.

This completes the proof.

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