Betti numbers of monomial curves*

Ignacio García-Marco†1, Philippe Gimenez‡2, and Mario González-Sánchez§2

[1]Instituto de Matemáticas y Aplicaciones (IMAULL), Sección de Matemáticas, Facultad de Ciencias, Universidad de La Laguna, 38200, La Laguna, Spain
[2]Instituto de Investigación en Matemáticas de la Universidad de Valladolid (IMUVA), Universidad de Valladolid, 47011 Valladolid, Spain

Abstract

In this work, we explore when the Betti numbers of the coordinate rings of a projective monomial curve and one of its affine charts are identical. Given an infinite field k and a sequence of relatively prime integers $a_0 = 0 < a_1 < \cdots < a_n = d$, we consider the projective monomial curve $\mathcal{C} \subset \mathbb{P}_k^n$ of degree d parametrically defined by $x_i = u^{a_i} v^{d-a_i}$ for all $i \in \{0, \ldots, n\}$ and its coordinate ring $k[\mathcal{C}]$. The curve $\mathcal{C}_1 \subset \mathbb{A}_k^n$ with parametric equations $x_i = t^{a_i}$ for $i \in \{1, \ldots, n\}$ is an affine chart of $\mathcal{C}$ and we denote by $k[\mathcal{C}_1]$ its coordinate ring. The main contribution of this paper is the introduction of a novel (Gröbner-free) combinatorial criterion that provides a sufficient condition for the equality of the Betti numbers of $k[\mathcal{C}]$ and $k[\mathcal{C}_1]$. Leveraging this criterion, we identify infinite families of projective curves satisfying this property.

Introduction

Let k be an infinite field, and $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ and $k[\mathbf{t}] := k[t_1, \ldots, t_m]$ be two polynomial rings over k. Given $\mathcal{B} = \{b_1, \ldots, b_n\} \subset \mathbb{N}^m$, a set of nonzero vectors, each element $b_i = (b_{i1}, \ldots, b_{im}) \in \mathbb{N}^m$ corresponds to the monomial $\mathbf{t}^{b_i} := t_1^{b_{i1}} \cdots t_m^{b_{im}} \in k[\mathbf{t}]$. The affine toric variety $X_\mathcal{B} \subset \mathbb{A}_k^n$ determined by $\mathcal{B}$ is the Zariski closure of the set given parametrically by $x_i = u_1^{b_{i1}} \cdots u_m^{b_{im}}$ for all $i = 1, \ldots, n$. Consider

$$\mathcal{S}_\mathcal{B} := \langle b_1, \ldots, b_n \rangle = \{\alpha_1 b_1 + \cdots + \alpha_n b_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{N}\} \subset \mathbb{N}^m ,$$

the affine monoid spanned by $\mathcal{B}$. The toric ideal determined by $\mathcal{B}$ is the kernel $I_\mathcal{B}$ of the k-algebra homomorphism $\varphi_\mathcal{B} : k[\mathbf{x}] \longrightarrow k[\mathbf{t}]$ induced by $x_i \mapsto \mathbf{t}^{b_i}$. Since k is infinite, one has that $I_\mathcal{B}$ is the vanishing ideal of $X_\mathcal{B}$ and, hence, the coordinate ring of $X_\mathcal{B}$ is (isomorphic to) the semigroup algebra $k[\mathcal{S}_\mathcal{B}] := \mathrm{Im}(\varphi_\mathcal{B}) \simeq k[\mathbf{x}]/I_\mathcal{B}$. The ideal $I_\mathcal{B}$ is an $\mathcal{S}_\mathcal{B}$-homogeneous binomial ideal, i.e., if one sets the $\mathcal{S}_\mathcal{B}$-degree of a monomial $\mathbf{x}^\alpha \in k[\mathbf{x}]$ as $\deg_{\mathcal{S}_\mathcal{B}}(\mathbf{x}^\alpha) := \alpha_1 b_1 + \cdots + \alpha_n b_n \in \mathcal{S}_\mathcal{B}$, then $I_\mathcal{B}$ is generated by $\mathcal{S}_\mathcal{B}$-homogeneous binomials. One can thus consider a minimal $\mathcal{S}_\mathcal{B}$-graded free resolution of $k[\mathcal{S}_\mathcal{B}]$ as $\mathcal{S}_\mathcal{B}$-graded $k[\mathbf{x}]$-module,

$$\mathcal{F} : 0 \longrightarrow F_p \longrightarrow \cdots \longrightarrow F_0 \longrightarrow k[\mathcal{S}_\mathcal{B}] \longrightarrow 0 .$$

The projective dimension of $k[\mathcal{S}_\mathcal{B}]$ is $\mathrm{pd}(k[\mathcal{S}_\mathcal{B}]) = \max\{i \mid F_i \neq 0\}$. The i-th Betti number of $k[\mathcal{S}_\mathcal{B}]$ is the rank of the free module F_i, i.e., $\beta_i(k[\mathcal{S}_\mathcal{B}]) = \mathrm{rank}(F_i)$; and the Betti sequence of $k[\mathcal{S}_\mathcal{B}]$ is

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†Email: iggarcia@ull.edu.es.
‡Email: pgimenez@uva.es.
§Email: mario.gonzalez.sanchez@uva.es. T. A. thanks financial support from European Social Fund, *Programa Operativo de Castilla y León*, and *Consejería de Educación de la Junta de Castilla y León*.

$(\beta_i(k[\mathcal{S}_\mathcal{B}])\,;\, 0 \leq i \leq \mathrm{pd}(k[\mathcal{S}_\mathcal{B}]))$. When the Krull dimension of $k[\mathcal{S}_\mathcal{B}]$ coincides with its depth as $k[\mathbf{x}]$-module, the ring $k[\mathcal{S}_\mathcal{B}]$ is said to be Cohen-Macaulay. By the Auslander-Buchsbaum formula, this is equivalent to $\mathrm{pd}(k[\mathcal{S}_\mathcal{B}]) = n - \dim(k[\mathcal{S}_\mathcal{B}])$. When $k[\mathcal{S}_\mathcal{B}]$ is Cohen-Macaulay, its (Cohen-Macaulay) type is the rank of the last nonzero module in the resolution, i.e., $\mathrm{type}(k[\mathcal{S}_\mathcal{B}]) := \beta_p(k[\mathcal{S}_\mathcal{B}])$ where $p = \mathrm{pd}(k[\mathcal{S}_\mathcal{B}])$.

Now consider $d \in \mathbb{Z}^+$ and $a_0 := 0 < a_1 < \cdots < a_n = d$ a sequence of relatively prime integers. Denote by $\mathcal{C}$ the projective monomial curve $\mathcal{C} \subset \mathbb{P}_k^n$ of degree d parametrically defined by $x_i = u^{a_i} v^{d-a_i}$ for all $i \in \{0, \ldots, n\}$, i.e., $\mathcal{C}$ is the Zariski closure of

$$\{(u^{a_0} v^{d-a_0} : \cdots : u^{a_i} v^{d-a_i} : \cdots : u^{a_n} v^{d-a_n}) \in \mathbb{P}_k^n \,|\, (u : v) \in \mathbb{P}_k^1\}.$$

Taking $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^2$ with $\mathbf{a}_i = (a_i, d - a_i)$ for all $i = 0, \ldots, n$, one has that $I_\mathcal{A}$ is the vanishing ideal of $\mathcal{C}$, and the coordinate ring of $\mathcal{C}$ is the two-dimensional ring $k[\mathcal{C}] = k[x_0, \ldots, x_n]/I_\mathcal{A}$, where $\mathcal{S} = \mathcal{S}_\mathcal{A}$ denotes the monoid spanned by $\mathcal{A}$. The projective monomial curve $\mathcal{C}$ is said to be arithmetically Cohen-Macaulay if the ring $k[\mathcal{C}]$ is Cohen-Macaulay.

The monomial projective curve $\mathcal{C}$ has two affine charts, $\mathcal{C}_1 = \{(u^{a_1}, \ldots, u^{a_n}) \in \mathbb{A}_k^n \,|\, u \in k\}$ and $\mathcal{C}_2 = \{(v^{d-a_0}, v^{d-a_1}, \ldots, v^{d-a_{n-1}}) \in \mathbb{A}_k^n \,|\, v \in k\}$, associated to the sequences $a_1 < \cdots < a_n$ and $d - a_{n-1} < \cdots < d - a_1 < d - a_0$, respectively. The second sequence is sometimes called the dual of the first one. Denote by $\mathcal{S}_1 := \mathcal{S}_{\mathcal{A}_1}$ the numerical semigroup generated by $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$. The vanishing ideal of $\mathcal{C}_1$ is $I_{\mathcal{A}_1} \subset k[x_1, \ldots, x_n]$, and hence, its coordinate ring is the one-dimensional ring $k[\mathcal{C}_1] = k[x_1, \ldots, x_n]/I_{\mathcal{A}_1}$. Moreover, $I_\mathcal{A}$ is the homogenization of $I_{\mathcal{A}_1}$ with respect to the variable x_0. Similarly, denoting by $\mathcal{S}_2 := \mathcal{S}_{\mathcal{A}_2}$ the numerical semigroup generated by $\mathcal{A}_2 := \{d - a_0, d - a_1, \ldots, d - a_{n-1}\}$, the vanishing ideal of $\mathcal{C}_2$ is $I_{\mathcal{A}_2} \subset k[x_0, \ldots, x_{n-1}]$, its coordinate ring is $k[\mathcal{C}_2] = k[x_0, \ldots, x_{n-1}]/I_{\mathcal{A}_2}$, and $I_\mathcal{A}$ is the homogenization of $I_{\mathcal{A}_2}$ with respect to x_n.

One has that $\beta_i(k[\mathcal{C}]) \geq \beta_i(k[\mathcal{C}_1])$ for all i, and the goal of this work is to understand when the Betti sequences of $k[\mathcal{C}]$ and $k[\mathcal{C}_1]$ coincide. A necessary condition is that $k[\mathcal{C}]$ is Cohen-Macaulay. Indeed, affine monomial curves are always arithmetically Cohen-Macaulay while projective ones may be arithmetically Cohen-Macaulay or not. Thus, $\mathrm{pd}(k[\mathcal{C}]) = \mathrm{pd}(k[\mathcal{C}_1])$ if and only if $\mathcal{C}$ is arithmetically Cohen-Macaulay. In Theorem 5, which is the main result of this work, we provide a combinatorial sufficient condition for having equality between the Betti sequences of $k[\mathcal{C}]$ and $k[\mathcal{C}_1]$ by means of the poset structures induced by $\mathcal{S}$ and $\mathcal{S}_1$ on the Apery sets of both $\mathcal{S}$ and $\mathcal{S}_1$. In Propositions 9 and 11, we use our main result to provide explicit families of curves where $\beta_i(k[\mathcal{C}]) = \beta_i(k[\mathcal{C}_1])$ for all i.

The motivation of this work comes from [7], where the authors obtain a sufficient condition in terms of Gröbner bases to ensure the equality of the Betti sequences.

The computations in the examples given in this paper are performed using Singular [4].

1 Apery sets and their poset structure

Let $d \in \mathbb{Z}^+$ and $a_0 := 0 < a_1 < \cdots < a_n = d$ be a sequence of relatively prime integers. For each $i = 0, \ldots, n$, set $\mathbf{a}_i := (a_i, d - a_i) \in \mathbb{N}^2$, and consider the three sets $\mathcal{A}_1 = \{a_1, \ldots, a_n\}$, $\mathcal{A}_2 = \{d, d - a_1, \ldots, d - a_{n-1}\}$ and $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\} \subset \mathbb{N}^2$. We denote by $\mathcal{C} \subset \mathbb{P}_k^n$ the projective monomial curve defined by $\mathcal{A}$ as defined in the introduction, and by $\mathcal{C}_1$ and $\mathcal{C}_2$ its affine charts. Consider $\mathcal{S}_1$ and $\mathcal{S}_2$ the numerical semigroups generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, and $\mathcal{S}$ the monoid spanned by $\mathcal{A}$ that we call the homogenization of $\mathcal{S}_1$ (with respect to d).

As already mentioned, $k[\mathcal{S}_1]$ and $k[\mathcal{S}_2]$ are always Cohen-Macaulay, while $k[\mathcal{C}]$ can be Cohen-Macaulay or not. There are many ways to determine when a projective monomial curve is arithmetically Cohen-Macaulay; see, e.g., [2, Cor. 4.2], [3, Lem. 4.3, Thm. 4.6] or [6, Thm. 2.6]. We give some of them in Proposition 1, but let us previously recall the notion of Apery set since it is involved in some of those charaterizations.

For $i = 1, 2$, the Apery set of $\mathcal{S}_i$ with respect to d is $\mathrm{Ap}_i := \{y \in \mathcal{S}_i \,|\, y - d \notin \mathcal{S}_i\}$. Since $\gcd(\mathcal{A}_1) = 1$, we know that Ap_i is a complete set of residues modulo d, i.e., $\mathrm{Ap}_1 = \{r_0 = 0, r_1, \ldots, r_{d-1}\}$ and $\mathrm{Ap}_2 = \{t_0 = 0, t_1, \ldots, t_{d-1}\}$ for some positive integers r_i and t_i such that $r_i \equiv t_i \equiv i \pmod d$ for all $i = 1, \ldots, d-1$. One can also define the Apery set of $\mathcal{S}$ as $\mathrm{AP}_{\mathcal{S}} := \{\mathbf{y} \in \mathcal{S} \,|\, \mathbf{y} - \mathbf{a}_0 \notin \mathcal{S}, \mathbf{y} - \mathbf{a}_n \notin \mathcal{S}\}$. Note that this set has at least d elements by [5, Lem. 2.5].

Proposition 1. *The following assertions are equivalent:*

(a) $\mathcal{C}$ *is arithmetically Cohen-Macaulay.*

(b) $\mathrm{AP}_{\mathcal{S}}$ *has exactly* d *elements.*

(c) $\mathrm{AP}_{\mathcal{S}} = \{(0,0)\} \cup \{(r_i, t_{d-i}) \,|\, 1 \leq i < d\}$.

(d) For all $i = 1, \ldots, d-1$, $(r_i, t_{d-i}) \in \mathcal{S}$. *In other words, if* $q_1 \in \mathrm{Ap}_1$, $q_2 \in \mathrm{Ap}_2$ *and* $q_1 + q_2 \equiv 0 \pmod d$*, then* $(q_1, q_2) \in \mathcal{S}$.

(e) If $\mathbf{s} \in \mathbb{Z}^2$ *satisfies* $\mathbf{s} + \mathbf{a}_0 \in \mathcal{S}$ *and* $\mathbf{s} + \mathbf{a}_n \in \mathcal{S}$*, then* $\mathbf{s} \in \mathcal{S}$.

In order to compare $\beta_i(k[\mathcal{C}])$ and $\beta_i(k[\mathcal{C}_1])$ for all i, we will relate in Theorem 5 the Apery sets Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ with the natural poset structure that both have and that we now define. For $i = 1, 2$, $(\mathrm{Ap}_i, \leq_i)$ is a poset, where $\leq_i$ is given by $y \leq_i z \iff z - y \in \mathcal{S}_i$. Similarly, $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ is a poset for $\leq_{\mathcal{S}}$ defined by $\mathbf{y} \leq_{\mathcal{S}} \mathbf{z} \iff \mathbf{z} - \mathbf{y} \in \mathcal{S}$.

Since $\mathcal{S} \subset \mathcal{S}_1 \times \mathcal{S}_2$, it follows that if $(y_1, y_2) \leq_{\mathcal{S}} (z_1, z_2)$, then $y_i \leq_i z_i$ for $i = 1, 2$. Using Proposition 1, one can prove that the poset structure of $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ is completely determined by those of $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{Ap}_2, \leq_2)$ when $\mathcal{C}$ is arithmetically Cohen-Macaulay.

Proposition 2. *If* $\mathcal{C}$ *is arithmetically Cohen-Macaulay, then for all* $(y_1, y_2), (z_1, z_2) \in AP_{\mathcal{S}}$,

$$(y_1, y_2) \leq_{\mathcal{S}} (z_1, z_2) \iff y_1 \leq_1 z_1 \text{ and } y_2 \leq_2 z_2.$$

Let us recall some notions about posets that will be needed in the sequel.

Definition 3. *Let* $(P, \leq)$ *be a finite poset.*

(a) For $y, z \in P$*, we say that* z *covers* y*, and denote it by* $y \prec z$*, if* $y < z$ *and there is no* $w \in P$ *such that* $y < w < z$.

(b) We say that P *is graded if there exists a function* $\rho : P \to \mathbb{N}$*, called rank function, such that* $\rho(z) = \rho(y) + 1$ *whenever* $y \prec z$.

As the following result shows, the poset $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ is always graded. Since $(\mathrm{Ap}_1, \leq_1)$ has a minimum, whenever it is graded, the corresponding rank function is completely determined by the value of the rank function in the minimum, which we will fix to be 0. In the following proposition, we characterize the covering relation in Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ and describe the rank functions of $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$, and of $(\mathrm{Ap}_1, \leq_1)$ when it is graded.

Proposition 4. *(a) If* $y, z \in \mathrm{Ap}_1$*, then* $y \prec_1 z$ *if and only if* $z = y + a_i$ *for some minimal generator* a_i *of* $\mathcal{S}_1$ *such that* $a_i \neq d$*. Therefore, if* Ap_1 *is graded and* $\rho_1 : \mathrm{Ap}_1 \to \mathbb{N}$ *denotes the rank function, for any* $y \in \mathrm{Ap}_1$*,* $\rho_1(y)$ *is the number of elements involved in any writing of* y *in terms of minimal generators of* $\mathcal{S}_1$.

(b) If $\mathbf{y} = (y_1, y_2)$*,* $\mathbf{z} = (z_1, z_2)$ *and* $\mathbf{y}, \mathbf{z} \in \mathrm{AP}_{\mathcal{S}}$*, then* $\mathbf{y} \prec_{\mathcal{S}} \mathbf{z}$ *if and only if* $\mathbf{z} = \mathbf{y} + \mathbf{a}_i$ *for some* $i \in \{1, \ldots, n-1\}$*. Therefore,* $\mathrm{AP}_{\mathcal{S}}$ *is graded by the rank function* $\rho : \mathrm{AP}_{\mathcal{S}} \to \mathbb{N}$ *defined by* $\rho(y_1, y_2) := (y_1 + y_2)/d$.

2 Betti numbers of affine and projective monomial curves

Recall that $I_{\mathcal{A}_1} \subset k[x_1, \ldots, x_n]$ is the vanishing ideal of $\mathcal{C}_1$ and $I_{\mathcal{A}} \subset k[x_0, \ldots, x_n]$ is the vanishing ideal of $\mathcal{C}$. When $\mathcal{C}$ is arithmetically Cohen-Macaulay, $\mathrm{pd}(k[\mathcal{C}]) = \mathrm{pd}(k[\mathcal{C}_1])$. Moreover, by Proposition 1, in this case, one has that $|\mathrm{AP}_{\mathcal{S}}| = |\mathrm{Ap}_1| = d$. The main result in this section is Theorem 5 where we give a sufficient condition in terms of the poset structures of the Apery sets Ap_1 and $\mathrm{AP}_{\mathcal{S}}$ for the Betti sequences of $k[\mathcal{C}_1]$ and $k[\mathcal{C}]$ to coincide.

Theorem 5. *If* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1)$, *then* $\beta_i(k[\mathcal{C}]) = \beta_i(k[\mathcal{C}_1])$ *for all* i.

Note that the converse of this result is far from being true, as shown in Example 6.

Example 6. *For the sequence* $1 < 2 < 4 < 8$, *one has that both* $k[\mathcal{C}_1]$ *and* $k[\mathcal{C}]$ *are complete intersections with Betti sequence* $(1, 3, 3, 1)$. *However, the posets* $(\mathrm{Ap}_1, \leq_1)$ *and* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ *are not isomorphic since* $\leq_1$ *is a total order on* Ap_1, *while* $\leq_{\mathcal{S}}$ *is not.*

In order to compare the two posets $\mathrm{AP}_{\mathcal{S}}$ and Ap_1, one can use the following result.

Proposition 7. *The following two claims are equivalent:*

(a) *The posets* $(\mathrm{Ap}_1, \leq_1)$ *and* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ *are isomorphic;*

(b) $k[\mathcal{C}]$ *is Cohen-Macaulay,* $(\mathrm{Ap}_1, \leq_1)$ *is graded, and* $\{a_1, \ldots, a_{n-1}\}$ *is contained in the minimal system of generators of* $\mathcal{S}_1$.

Note that Ap_1 can be a graded poset even if $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are not isomorphic as the following example shows.

Example 8. *For the sequence* $a_1 = 5 < a_2 = 11 < a_3 = 13$, *the Apery set of the numerical semigroup* $\mathcal{S}_1 = \langle a_1, a_2, a_3 \rangle$ *is* $\mathrm{Ap}_1 = \{0, 27, 15, 16, 30, 5, 32, 20, 21, 22, 10, 11, 25\}$. *This Apery set is graded with the rank function* $\rho_1 : \mathcal{S}_1 \to \mathbb{N}$ *defined below (see Figure 1):*

- $\rho_1(0) = 0$,
- $\rho_1(5) = \rho(11) = 1$,
- $\rho_1(10) = \rho_1(16) = \rho_1(22) = 2$,
- $\rho_1(15) = \rho_1(21) = \rho_1(27) = 3$,
- $\rho_1(20) = \rho_1(32) = 4$,
- $\rho_1(25) = 5$,
- $\rho_1(30) = 6$.

Moreover, since $\mathrm{AP}_{\mathcal{S}}$ *has 16 elements,* $k[\mathcal{C}]$ *is not Cohen-Macaulay, and hence* $(\mathrm{Ap}_1, \leq_1)$ *and* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ *are not isomorphic by Proposition 7.*

3 Examples of application

In Propositions 9 and 11, we provide some sequences $a_1 < \cdots < a_n$ for which the condition in Theorem 5 is satisfied. Let us start with arithmetic sequences, i.e., sequences $a_1 < \cdots < a_n$ such that $a_i = a_1 + (i-1)e$ for some positive integer e with $\gcd(a_1, e) = 1$. For this family, we refine [7, Cor. 4.2] that considers $a_1 > n - 1$.

Proposition 9. *Let* $a_1 < \ldots < a_n$ *be an arithmetic sequence of relatively prime integers. Then,* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1)$ *if and only if* $a_1 > n - 2$. *Therefore, if* $a_1 > n - 2$, *the Betti sequences of* $k[\mathcal{C}_1]$ *and* $k[\mathcal{C}]$ *coincide.*

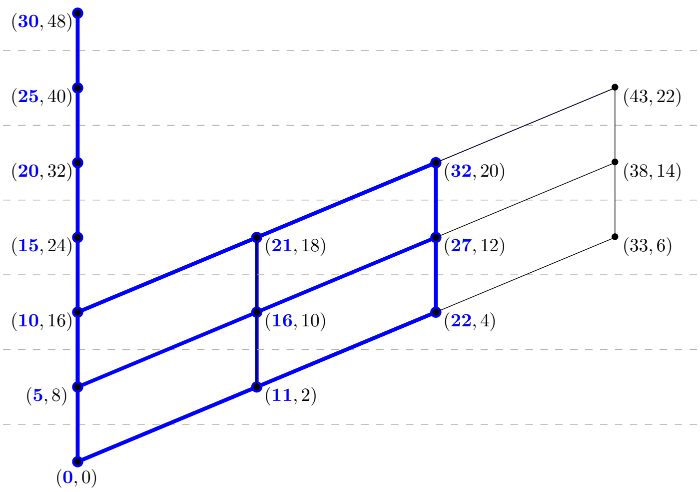

Figure 1: The posets $(\mathrm{Ap}_1, \leq_1)$ (in blue) and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ (in black) for $\mathcal{S}_1 = \langle 5, 11, 13 \rangle$.

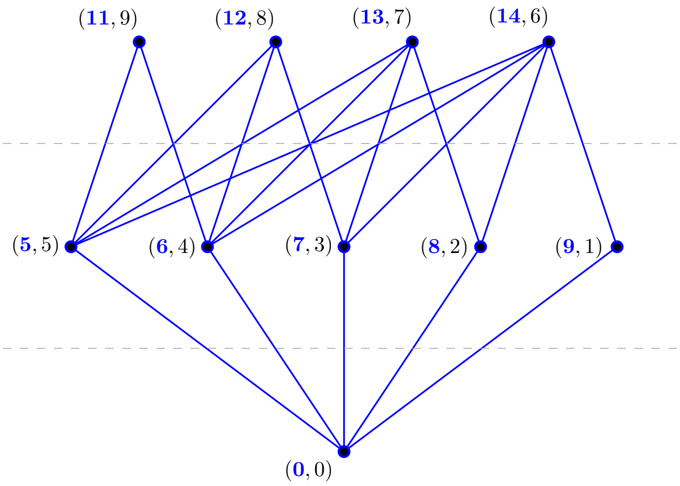

Figure 2: The posets $(\mathrm{Ap}_1, \leq_1)$ (in blue) and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ (in black) for $\mathcal{S}_1 = \langle 5, 6, 7, 8, 9, 10 \rangle$.

Example 10. *For the sequence* $5 < 6 < 7 < 8 < 9 < 10$*, one has that* $a_1 = 5 > 4 = n - 2$*. Therefore, the Apery sets* $(\mathrm{Ap}_1, \leq_1)$ *and* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ *are isomorphic. Hence, by Theorem 5, the Betti sequences of* $k[\mathcal{C}_1]$ *and* $k[\mathcal{C}]$ *coincide. One can check that both are* $(1, 11, 30, 35, 19, 4)$*. The posets* $(\mathrm{Ap}_1, \leq_1)$ *and* $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ *in this example are shown in Figure 2.*

In [1, Sect. 6], the authors studied the canonical projections of the projective monomial curve $\mathcal{C}$ defined by an arithmetic sequence $a_1 < \cdots < a_n$ of relatively prime integers, i.e., the curve $\pi_r(\mathcal{C})$ obtained as the Zariski closure of the image of $\mathcal{C}$ under the r-th canonical projection $\pi_r : \mathbb{P}_k^n \dashrightarrow \mathbb{P}_k^{n-1}$, $(p_0 : \cdots : p_n) \mapsto (p_0 : \cdots : p_{r-1} : p_{r+1} : \cdots : p_n)$. We know that $\pi_r(\mathcal{C})$ is the projective monomial curve associated to the sequence $a_1 < \cdots < a_{r-1} < a_{r+1} < \cdots < a_n$.

In Proposition 11, for any $r \in \{2, \ldots, n-1\}$, we consider $\mathcal{A}_1 = \{a_1, \ldots, a_n\} \setminus \{a_r\}$, the numerical semigroup $\mathcal{S}_1 = \mathcal{S}_{\mathcal{A}_1}$, and its homogenization $\mathcal{S}$, and we characterize when the posets $(\mathrm{Ap}_1, \leq_1)$ and $(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}})$ are isomorphic.

Proposition 11. *Consider* $a_1 < \ldots < a_n$ *an arithmetic sequence of relatively prime integers with* $n \geq 4$*, and take* $r \in \{2, \ldots, n-1\}$*. Set* $\mathcal{A}_1 := \{a_1, \ldots, a_n\} \setminus \{a_r\}$*, and let* $\mathcal{S}_1$ *be the numerical semigroup generated by* $\mathcal{A}_1$*, and* $\mathcal{S}$ *its homogenization. Then,*

$$(\mathrm{AP}_{\mathcal{S}}, \leq_{\mathcal{S}}) \simeq (\mathrm{Ap}_1, \leq_1) \iff \begin{cases} a_1 > n-2 \text{ and } a_1 \neq n, & \text{if } r = 2, \\ a_1 \geq n \text{ and } r \leq a_1 - n + 1, & \text{if } 3 \leq r \leq n-2, \\ a_1 \geq n-2, & \text{if } r = n-1. \end{cases}$$

Consequently, if the previous condition holds, then $\beta_i(k[\mathcal{C}_1]) = \beta_i(k[\mathcal{C}])$*, for all* i*.*

Example 12. *For the sequence* $9 < 10 < 11 < 12 < 13$*, the Betti sequences of* $k[\mathcal{C}_1]$ *and* $k[\mathcal{C}]$ *coincide by Proposition 9. Indeed, it is* $(1, 10, 20, 15, 4)$ *for both curves. The parameters of this arithmetic sequence are* $a_1 = 9$*,* $e = 1$ *and* $n = 5$*. Hence, the Betti sequences of* $k[\pi_r(\mathcal{C}_1)]$ *and* $k[\pi_r(\mathcal{C})]$ *coincide for* $r = 2, 3, 4$ *by Proposition 11. One can check that the Betti sequence of* $k[\pi_2(\mathcal{C})]$ *and* $k[\pi_4(\mathcal{C})]$ *is* $(1, 5, 6, 2)$*, and the Betti sequence of* $k[\pi_3(\mathcal{C})]$ *is* $(1, 8, 12, 5)$*.*

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