Extended abstract of Regular polytopes, sphere packings and Apollonian sections*

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Abstract

In this paper, we explore the geometry and the arithmetic of a family of polytopal sphere packings induced by regular polytopes in any dimension. We prove that every integral polytope is crystallographic and we show that there are 11 crystallographic regular polytopes in any dimension. After introducing the notion of Apollonian section, we determine which Platonic crystallographic packings emerge as cross sections of the Apollonian arrangements of the regular 4-polytopes. Additionally, we compute the Möbius spectrum of every regular polytope.

1 Introduction

Apollonian circle packings and their generalizations are currently active areas of research in geometric number theory [9, 10, 2]. In dimension 2, some variants of integral Apollonian packings have been explored by substituting the building block with a different circle packing modeled on a polyhedron [8, 22, 23, 3, 5, 14]. While every polyhedron can be employed to construct a packing, not all of them admit an integral structure like the Apollonian one. A fundamental question regarding the determination of which polyhedra are *integral* in this sense is still wide open [12, 5].

Similarly, in dimension 3, a family of crystallographic/Apollonian-like sphere packings arise by iteratively reflecting an initial sphere packing modeled on a 4-polytope as in Figure 1. Integral crystallographic packings modeled on the 4-simplex [20, 11] and the 4-cross polytope [13, 7, 19, 16] have been extensively studied. Unlike polyhedra, not every 4-polytope is *crystallographic*, in the sense that it serves as a suitable model for a crystallographic packing. In this paper, we delve into the crystallography and the integrality of regular polytopes in any dimension.

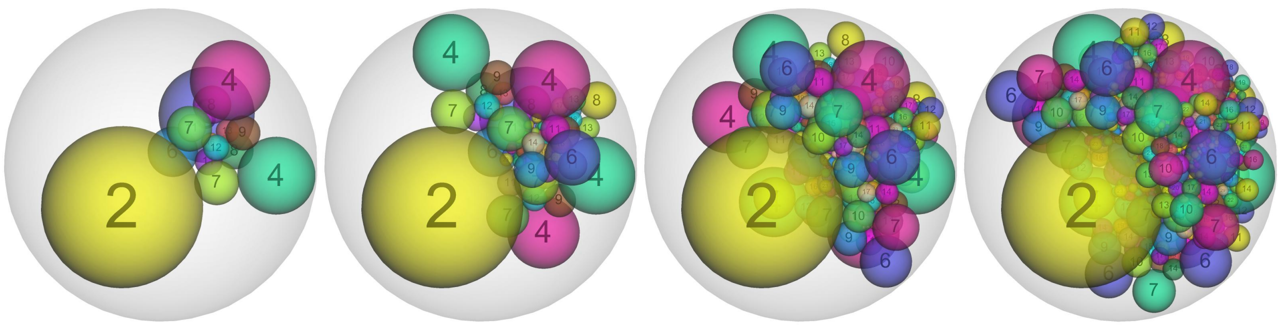

Figure 1: An integral hypercubic crystallographic packing after 0, 1, 2 and 3 iterations. The labels are the *bends* (reciprocal of the radii) of the spheres.

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2 Preliminaries on sphere packings and edge-scribable polytopes

An *oriented hypersphere*, or simply *sphere*, of $\widehat{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$, is the image of a spherical cap of $\mathbb{S}^d$ under the stereographic projection. Every sphere S is uniquely defined by its center $c \in \widehat{\mathbb{R}^d}$ and its bend $b \in \mathbb{R}$ (the recripocal of the *oriented* radius), or if S is a half-space, by its normal vector $\widehat{n} \in \mathbb{S}^{d-1}$ pointing to the interior and the signed distance $\delta \in \mathbb{R}$ between its boundary and the origin. The *inversive coordinates* of S are represented by the $(d+2)$-dimensional real vector

$$\mathbf{i}(S) = \begin{cases} \left(bc, \frac{\overline{b}-b}{2}, \frac{\overline{b}+b}{2}\right)^T & \text{if } b \neq 0, \\ \\ (\widehat{n}, \delta, \delta)^T & \text{otherwise} \end{cases} \tag{1}$$

where $\overline{b} = b\|c\|^2 - \frac{1}{b}$ is the *co-bend* of S. The co-bend is the bend of S after inversion through the unit sphere. The *inversive product* of two spheres S, S' of $\widehat{\mathbb{R}^d}$ is the real value

$$\langle S, S' \rangle = \mathbf{i}(S)^T \mathbf{Q}_{d+2} \mathbf{i}(S) \tag{2}$$

where $\mathbf{Q}_{d+2}$ is the diagonal matrix $\operatorname{diag}(1, \dots, 1, -1)$ of size $d+2$. The inversive product encodes the relative position of two spheres S and S' according to the following criteria:

$$\langle S, S' \rangle \begin{cases} < -1 & \text{if } S \cap S' = \emptyset, \\ = -1 & \text{if } \partial S \text{ and } \partial S' \text{ are tangent and } \operatorname{int}(S) \cap \operatorname{int}(S') = \emptyset, \\ = 1 & \text{if } \partial S \text{ and } \partial S' \text{ are tangent and } S \subseteq S' \text{ or } S' \subseteq S, \\ > 1 & \text{if } \partial S \cap \partial S' = \emptyset \text{ and } S \subset S' \text{ or } S' \subset S. \end{cases} \tag{3}$$

An arrangement of spheres $\mathcal{S}$ in $\widehat{\mathbb{R}^d}$, possible infinite, is a *packing* if their interiors are mutually disjoint. The group of Möbius transformations of $\widehat{\mathbb{R}^d}$ preserves the inversive product and acts linearly on the inversive coordinates as an orthogonal subgroup of $\mathrm{SL}_{d+2}(\mathbb{R})$ with respect to $\mathbf{Q}_{d+2}$.

For every $d \geq 1$, we denote the *polar* of a subset $X \subset \mathbb{R}^d$ by $X^* = \{u \in \mathbb{R}^d \mid \langle u, v \rangle \leq 1 \text{ for all } v \in X\}$. The *stereographic sphere* of a point $v \in \mathbb{R}^d$ *outside* $\mathbb{S}^{d-1}$ (i.e. with $\|v\| > 1$) is the sphere S_v of $\widehat{\mathbb{R}^{d-1}}$ obtained by the stereographic projection of the spherical cap $\{-v\}^* \cap \mathbb{S}^{d-1}$. For any d-polytope $\mathcal{P}$ with vertices outside the unit sphere, the *(sphere) arrangement projection* of $\mathcal{P}$ is defined as the arrangement $\mathcal{S}_\mathcal{P}$ of the stereographic spheres of the vertices of $\mathcal{P}$.

A d-polytope is termed *edge-scribed* if its edges are tangent to the unit sphere [6]. If, in addition, the barycenter of the contact points is the origin, it is referred to as *canonical* [24]. A d-polytope is considered *edge-scribable* if it admits an edge-scribed realization [6]. In dimension $d \geq 3$, all the edge-scribed realizations of an edge-scribable d-polytope $\mathcal{P}$ are equivalent up to Möbius transformations to a unique canonical realization $\mathcal{P}_0$ (see [21, 14] for more details).

The arrangement projection of an edge-scribed polytope is a packing. Reciprocally, we say that a sphere packing $\mathcal{S}_\mathcal{P}$ in $\widehat{\mathbb{R}^d}$ with $d \geq 2$, is *polytopal* if there is an edge-scribable $(d+1)$-polytope $\mathcal{P}$ and a Möbius transformation μ such that $\mathcal{S}_\mathcal{P} = \mu \cdot \mathcal{S}_{\mathcal{P}_0}$. The combinatorial structure of $\mathcal{S}_\mathcal{P}$ is encoded by the corresponding edge-scribable polytope $\mathcal{P}$. The vertices and the edges of $\mathcal{P}$ are in bijection to the spheres and the tangency relations of $\mathcal{S}_\mathcal{P}$. The facets of $\mathcal{P}$ correspond to the *dual spheres* of $\mathcal{S}_\mathcal{P}$ which are the spheres forming the *dual arrangement* $\mathcal{S}^*_\mathcal{P} := \mu \cdot \mathcal{S}_{\mathcal{P}^*_0}$. The *Apollonian arrangement* of $\mathcal{S}_\mathcal{P}$ is defined as the orbit space $\mathscr{P}(\mathcal{S}_\mathcal{P}) := \langle \mathcal{S}^*_\mathcal{P} \rangle \cdot \mathcal{S}_\mathcal{P}$ where $\langle \mathcal{S}^*_\mathcal{P} \rangle$ denotes the group generated by inversions through the dual spheres. We denote by $\mathscr{P}_{\{p_1,\dots,p_d\}}$ the Apollonian arrangement of a regular polytope with Schläfli symbol $\{p_1, \dots, p_d\}$.

2.1 Crystallographic polytopes

In dimension 2, the Apollonian arrangements of 3-polytopes are packings, but this is not true in general [14]. In higher dimensions, Apollonian arrangements which are packings belong to the family of *crystallographic sphere packings* introduced by Kontorovich and Nakamura in [12]. These are dense infinite sphere packings obtained as the orbit space $\mathscr{P} = \langle \widetilde{\mathcal{S}} \rangle \cdot \mathcal{S}$, where $\mathcal{S}$ is a finite sphere packing called the *cluster*, $\langle \widetilde{\mathcal{S}} \rangle$ is a geometrically finite subgroup of the group of Möbius transformations generated by the inversions through a finite arrangement of spheres $\widetilde{\mathcal{S}}$, called the *co-cluster*, satisfying that every sphere of $\mathcal{S}$ is disjoint, tangent or orthogonal to every sphere of $\widetilde{\mathcal{S}}$.

Definition 1. *For every $d \geq 3$, an edge-scribable d-polytope $\mathcal{P}$ is crystallographic if any Apollonian arrangement $\mathscr{P}(\mathcal{S}_{\mathcal{P}}) = \langle \mathcal{S}^*_{\mathcal{P}} \rangle \cdot \mathcal{S}_{\mathcal{P}}$ is a sphere packing in dimension $d-1$.*

Crystallographic polytopes exist only in dimension $3 \leq d \leq 19$ [2]. From a Boyd's remark in [4], we have that an edge-scribable polytope $\mathcal{P}$ is crystallographic when the dihedral angles of $\mathcal{P}$, viewed as an hyperideal hyperbolic polytope, satisfy the *crystallographic restriction.* This restriction dictates that the periode of every rotation obtained as the product of two reflections through the facets is either $2, 3, 4, 6, \infty$, imposing a condition on the dihedral angles. On the other hand, the dihedral angle α of two adjacents facets f and f' of $\mathcal{P}$ is equal to the *intersection angle* of the corresponding dual spheres of $S_f, S_{f'} \in \mathcal{S}^*_{\mathcal{P}}$, as defined in [15]. This angle can be computed from their inversive product by $\langle S_f, S_{f'} \rangle = \cos(\alpha)$. Therefore, the crystallographic restriction can be reformuled in terms of the inversive product of the dual spheres, as described in Lemma 2.

Lemma 2. *For any $d \geq 3$, an edge-scribable d-polytope $\mathcal{P}$ is crystallographic if and only if for any two dual spheres S_f, $S_{f'}$ of a polytopal sphere packing $\mathcal{S}_{\mathcal{P}}$, we have $|\langle S_f, S_{f'} \rangle| \in \{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\} \cup [1, \infty)$.*

2.2 Integral polytopes

In [12], Kontorovich and Nakamura defined a 3-polytope $\mathcal{P}$ to be *integral*[1] if there is a crystallographic circle packing modeled on $\mathcal{P}$ where the bends of the spheres are all integers. The fundamental question regarding the determination of which 3-polytopes are integral is still wide open. In [5], Chait-Roth, Cui, and Stier studied the integral 3-polytopes with few vertices. Based on previous works of Nakamura and Kontorovich, they gave the following enumeration of the integral uniform 3-polytopes.

Theorem 3 (Th. 26 [5])**.** *There are only 8 integral uniform 3-polytopes: the tetrahedron, the octahedron, the cube, the cuboctahedron, the truncated tetrahedron, the truncated octahedron, the 3-prism and the 6-prism.*

In higher dimensions, the previous definition of integral 3-polytope can be naturally extended for any edge-scribable polytope.

Definition 4. *For any $d \geq 3$, an edge-scribable d-polytope $\mathcal{P}$ is integral if it admits an Apollonian arrangement $\mathscr{P}(\mathcal{S}_{\mathcal{P}})$ where the bends of the spheres are in $\mathbb{Z}$.*

A priori, an edge-scribable polytope might be integral and non-crystallographic, meaning that it could admit an Apollonian arrangement where the bends of the spheres are integers and the spheres overlap. Indeed, this is the case if we adapt the definition of integral polytope for number rings other than $\mathbb{Z}$. For instance, the 600-cell is integral in $\mathbb{Z}[\varphi]$, but is not crystallographic (see Figure 2).

[1] This definition of *integral polytope* differs from the one commonly employed in combinatorics, which involves polytopes with integer vertex coordinates, also known as *lattice polytopes*.

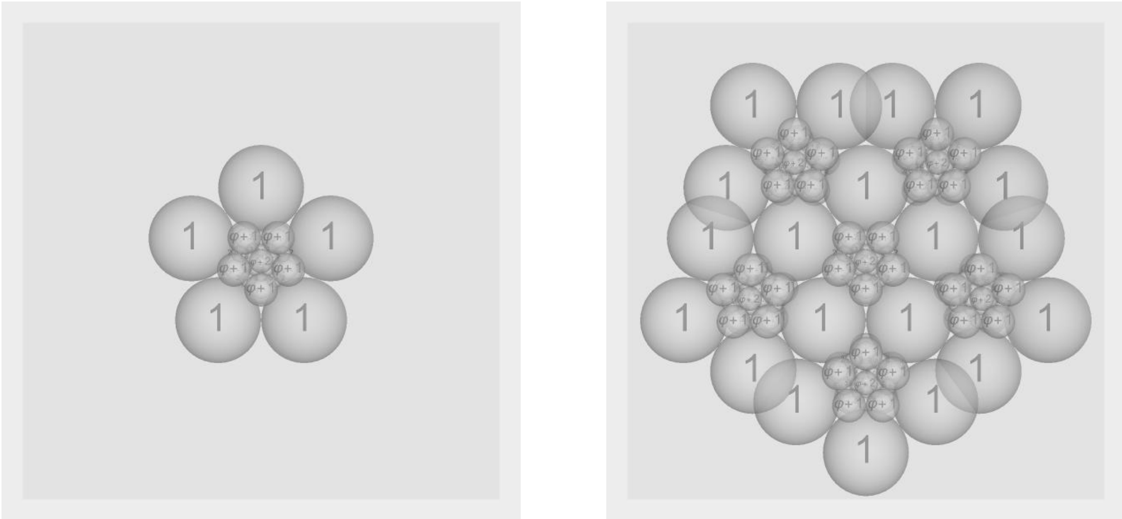

Figure 2: (Left) A polytopal sphere packing modeled on the 600-cell labelled with the bends; (right) the first reflections of its Apollonian arrangement which is integral in $\mathbb{Z}[\varphi]$ and is not a packing.

3 Main results

3.1 The relation between crystallography and integrality

In this paper, we prove the following condition for determining the integrality of edge-scribable polytopes.

Lemma 5. *For any $d \geq 3$, if an edge-scribable d-polytope $\mathcal{P}$ is integral, then for any two dual spheres S_f, $S_{f'}$ of any polytopal sphere packing $\mathcal{S}_\mathcal{P}$, we have $|\langle S_f, S_{f'}\rangle| \in \{\frac{\sqrt{n}}{2} \mid n \in \mathbb{N}\}$.*

With this lemma we can easily identify a mistake in the list of the integral uniform 3-polytopes of Chait-Roth, Cuit and Stier (Th. 3): the 6-prism is not integral, since it contains two dual spheres whose inversive product is $-5/3 \notin \{\pm\frac{\sqrt{n}}{2}\}_{n\in\mathbb{N}}$. Another straightforward consequence follows from Lemmas 2 and 5, and gives us the relation between crystallographic and integral polytopes in higher dimensions.

Theorem 6. *Every integral polytope is crystallographic.*

In the case of regular polytopes, we have the following.

Theorem 7. *For every $d \geq 3$, the only crystallographic regular d-polytopes are:*

($d = 3$) *the five Platonic solids,*

($d = 4$) *all the regular 4-polytopes except the 600-cell,*

($d = 6$) *the 6-cross polytope.*

Moreover, all these are integral except the icosahedron, the dodecahedron and the 120-cell which are integral in $\mathbb{Z}[\varphi]$.

3.2 Apollonian sections

The study of cross-sections is a classic approach for extracting patterns of crystallographic sphere packings [4, 1]. In this paper, we introduce an algebraic tool called *Apollonian section* which proves to be useful for identifing which Platonic crystallographic circle packings emerge as cross-sections of the Apollonian arrangements of the regular 4-polytopes.

Theorem 8. *There are the following relations between the Apollonian arrangements of the regular d-polytopes for $d = 3, 4$:*

$$\begin{aligned} \mathscr{P}_{\{3,3\}} &\subset \mathscr{P}_{\{3,3,3\}}, \\ \mathscr{P}_{\{3,3\}}, \mathscr{P}_{\{3,4\}}, \mathscr{P}_{\{4,3\}} &\subset \mathscr{P}_{\{3,3,4\}}, \\ \mathscr{P}_{\{4,3\}} &\subset \mathscr{P}_{\{4,3,3\}}, \\ \mathscr{P}_{\{3,4\}}, \mathscr{P}_{\{4,3\}} &\subset \mathscr{P}_{\{3,4,3\}}, \\ \mathscr{P}_{\{3,3\}}, \mathscr{P}_{\{3,5\}} &\subset \mathscr{P}_{\{3,3,5\}}, \\ \mathscr{P}_{\{5,3\}} &\subset \mathscr{P}_{\{5,3,3\}}, \end{aligned} \tag{4}$$

where "$\mathscr{P}_{\{p,q\}} \subset \mathscr{P}_{\{r,s,t\}}$" means that $\mathscr{P}_{\{p,q\}}$ can be obtained as a cross-section of $\mathscr{P}_{\{r,s,t\}}$.

Some of these cross-sections have been used as a geometric framework for obtaining results in geometric knot theory, as discussed in [16]. Another important feature of this approach is that it enable us to determine whether a cross-section preserves integrality.

Corollary 9. *Every integral Platonic crystallographic circle packing can be obtained as a cross-section of an integral regular crystallographic sphere packing.*

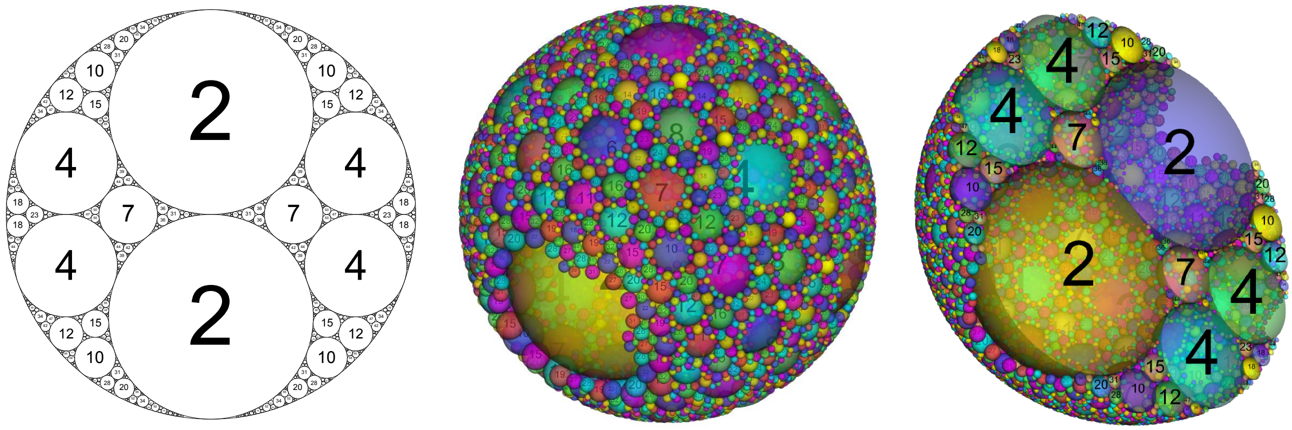

Figure 3: (Left) An integral octahedral crystallographic circle packing $\mathscr{P}_{\{3,4\}}$ obtained as a cross-section (right) of an integral orthoplicial crystallographic sphere packing $\mathscr{P}_{\{3,3,4\}}$ (center).

3.3 The Möbius spectrum of the regular polytopes

In [14], Ramírez Alfonsín and the author introduced a spectral invariant of every edge-scribable d-polytope $\mathcal{P}$ with $d \geq 3$ called the *Möbius spectrum* $\mathfrak{M}(\mathcal{P})$. This is defined as the multiset of the eigenvalues of the Gramian of any polytopal sphere packing $\mathcal{S}_{\mathcal{P}}$. Due to the Möbius uniqueness of edge-scribable polytopes, $\mathfrak{M}(\mathcal{P})$ does not depend on the packing. It is currently unknown whether there exist two combinatorially different edge-scribable polytopes with the same Möbius spectrum. In this paper, we compute the Möbius spectrum of every regular polytope $\mathcal{P}$ in terms of the number of vertices and another geometric invariant called the *canonical length* $\ell_{\mathcal{P}}$, defined as half the edge-length of a canonical realization of $\mathcal{P}$.

Theorem 10. *For any $d \geq 3$, the Möbius spectrum of every regular d-polytope $\mathcal{P}$ with n vertices is*

$$\mathfrak{M}(\mathcal{P}) = (-n\ell_{\mathcal{P}}^{-2}, \frac{n}{d}(1 + \ell_{\mathcal{P}}^{-2})^{(d)}, 0^{(n-d-1)}). \tag{5}$$

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