Separating cycle systems*

Fábio Botler†[1] and Tássio Naia‡[2]

[1]Instituto de Matemática e Estatística, São Paulo, Brazil
[2]Centre de Recerca Matemàtica, 2.7182 Sabadell, Spain

Abstract

A *separating system* of graph G is a family $\mathcal{F}$ of subgraphs of G such that, for all distinct edges $e, f \in E(G)$, some element in $\mathcal{F}$ contains e but not f. Recently, it has been shown that every n-vertex graph admits a separating system of paths of size $\mathrm{O}(n)$ [*Separating the edges of a graph by a linear number of paths*, M. Bonamy, F. Botler, F. Dross, T. Naia, J. Skokan. Advances in Combinatorics, October 2023]. This result improved an almost linear bound of $\mathrm{O}(n \log^\star n)$ found by Letzter in 2022, and settled a conjecture independently posed by Balogh, Csaba, Martin, and Pluhár and by Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan. We extend this result, showing that every n-vertex graph admits a separating system consisting of $O(n)$ edges and cycles.

1 Introduction

Given a set Ω and a family $\mathcal{F} \in 2^\Omega$ of subsets of Ω, we say that $\mathcal{F}$ *separates* Ω if for all distinct $\omega, \rho \in \Omega$ there exist $A_\omega, A_\rho \in \mathcal{F}$ such that $A_\omega \cap \{\omega, \rho\} = \{\omega\}$ and $A_\rho \cap \{\omega, \rho\} = \{\rho\}$. The study of separating systems dates back to the work of Rényi in 1961 [9]. The particular setting where $\Omega = E(G)$ is the edge set of a graph G and only certain subgraphs are allowed in $\mathcal{F}$ has also been investigated multiple times in the Computer Science literature, in the context of fault detection in networks (see, e.g., [5, 6] and the references therein). A generic problem in the area is the following.

Question 1. Let $\mathcal{G}$ be a (possibly infinite) family of graphs, and let H be an n-vertex graph. What is the smallest size of a collection $\mathcal{F} \subseteq \mathcal{G}$ of H-subgraphs such that $\{E(H) : H \in \mathcal{F}\}$ separates $E(H)$?

A *separating system* of a graph G is a collection of G-subgraphs such that their edge sets separate $E(G)$. Recently, Bonamy, Dross, Skokan and the two authors showed that every n-vertex graph admits a separating system consisting of at most $19n$ paths [2], improving a previous bound of $\mathrm{O}(n \log^\star n)$ found by Letzter in 2022 [7], and settling a conjecture independently posed by Balogh, Csaba, Martin, and Pluhár [1] and by Falgas-Ravry, Kittipassorn, Korándi, Letzter, and Narayanan [4].

A natural follow-up question is to ask whether every graph G admits a *cycle separating system* of size $\mathrm{O}\big(|V(G)|\big)$, that is, a collection of cycles and edges of G which separate $E(G)$. (Note that cycles alone are not enough in general, since G might contain a cycle-free component.) This question was independently posed by Girão and Pavez-Signé[1]. Here we answer their question in the affirmative.

*This research has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brazil – CAPES – Finance Code 001. This work is also supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). F. Botler was partially supported by CNPq (Proc. 304315/2022-2), FAPERJ (Proc. 201.334/2022), and CAPES (Proc. 88881.878881/2023-01). T. Naia was partially supported by the Grant PID2020-113082GB-I00 funded by MICIU/AEI/10.13039/501100011033.

†Email: fbotler@ime.usp.br

‡Email: tnaia@member.fsf.org

[1]Personal communication.

Theorem 2. *Every graph on n vertices admits a separating cycle system of size $41n$.*

Note that any cycle separating system of K_n contains at least $(n-1)/2$ elements, since each of the $\binom{n}{2}$ edges must be covered and any cycle contains at most n edges, so the bound in Theorem 2 is optimal apart from the leading constant. We do not believe that 41 is the correct multiplicative constant, and we wonder whether every graph of order n admits a separating cycle system of order $n + o(n)$.

Our proof uses a combination of properties of Pósa's rotation-extension method, a covering result due to Pyber, combined with algebraically-constructed edge covers of Hamiltonian graphs.

1.1 Pósa rotation-extension.

Given a graph G and $S \subseteq V(G)$, we denote by $N_G(S)$ the set of vertices in $V(G) \setminus S$ which are adjacent (in G) to some vertex in S. We omit subscripts when clear from the context. Let $P = u \cdots v$ be a path from u to v. If $x \in V(P)$ is a neighbor of u in G and x^- is the vertex preceding x in P, then $P' = P - xx^- + ux$ is a path in G for which $V(P') = V(P)$. We say that P' *has been obtained from* P by an *elementary exchange* fixing v (see Figure 1). A path obtained from P by a (possibly empty) sequence of elementary exchanges fixing v is said to be a path *derived* from P. The set of endvertices of paths derived from P distinct from v is denoted by $S_v(P)$. Since all paths derived from P have the same vertex set as P, we have $S_v(P) \subseteq V(P)$. When P is a longest path ending at v, we obtain the following (for a proof see [2]).

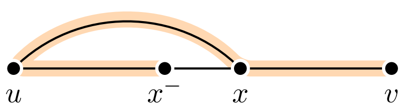

Figure 1: a path (highlighted) obtained by an elementary exchange fixing v.

Lemma 3 ([3])**.** *If $P = u \cdots v$ is a longest path of a graph G, then $\left|N_G(S)\right| \leqslant 2\left|S_v(P)\right|$.*

We also use the following property of Pósa rotations.

Lemma 4. *If $P = u \cdots v$ be a longest path of a graph G and $S = S_v(P)$, then G contains a subgraph C which is either and edge or a cycle and moreover $S \cup N(S) \subseteq C$.*

Proof. Consider the vertex $z \in V(P) \cap N(S)$ which lies closest to v in P, and let $P' = u' \cdots v$ be a path obtained from P by elementary exchanges fixing v so that P' starts with a neighbor u' of z. Note that C is an edge when $S = \{u\}$ and $|N(S)| = 1$. Since the section $P[z, v]$ of P from z to v intersects $S \cup N(S)$ precisely in v, and $P' \cup u'z$ has at most one cycle, we conclude that $C = (P' + uv) \setminus E(P[w, v])$ is either an edge or a cycle that contains $S \cup N(S)$ $\square$

2 Separating into cycles

For the sake of clarity, we make no attempt to optimize multiplicative constants in the argument. This allows us to better highlight its main ideas. It also seems unlikely that the optimal multiplicative constant can be reached by this approach alone.

The following theorem of Pyber is useful in our proof.

Theorem 5 (Pyber [8])**.** *Every graph G contains $|V(G)| - 1$ cycles and edges covering $E(G)$.*

Given a graph G, a collection $\mathcal{J}$ of subgraphs of G, and e, $f \in E(G)$, we say that $\mathcal{J}$ *separates* e *from* f if there exists $J \in \mathcal{J}$ such that $E(J) \cap \{e, f\} = \{e\}$. Similarly, given $\mathcal{E}$, $\mathcal{F} \subseteq E(G)$, we say that $\mathcal{J}$ *separates* $\mathcal{E}$ *from* $\mathcal{F}$ if $\mathcal{J}$ separates e from f for all distinct $e \in \mathcal{E}$ and $f \in \mathcal{F}$.

Proof of Theorem 2. We proceed by induction on n. Let G be a graph with n vertices. If G is empty, the result trivially holds. Let $P = u \cdots v$ be a longest path of G and let $S = S_v(P)$. By Lemma 4, there exists $C \subseteq G$ which is either an edge or a cycle and which contains $S \cup N(S)$.

Let H be the subgraph of G induced by the edges incident to at least a vertex in S, let $h = |V(H)|$, and let $G' = G \setminus S$. Then $G = H \cup G'$ and $V(H) = S \cup N(S)$, so $h \leqslant 3|S|$ by Lemma 3.

Note that S is not empty (because G is not empty). By the induction hypothesis, there is a cycle separating system $\mathcal{C}'$ of G' of size at most $41(n - |S|)$. Note that $\mathcal{C}'$ separates $E(G')$ from $E(H)$. In what follows, we construct a set $\mathcal{C}$ of at most $41|S|$ edges and cycles which separates $E(H)$ from $E(G)$, i.e., separates edges in H and also separates $E(H)$ from $E(G')$. This set $\mathcal{C}$ is the union of three collections of cycles and edges ($\mathcal{D}, \mathcal{E}$ and $\mathcal{H}$) which we next define.

Let $\mathcal{D}$ be a collection of at most $h-1 \leqslant 3|S|-1$ edges and cycles which covers $E(H) \setminus E(C)$ (such $\mathcal{D}$ exists by Lemma 5), and let $\mathcal{E} = E(C) \cap E(H)$ be the collection of edges of C which contain a vertex in S. Note that $|\mathcal{E}| \leqslant 2|S|$, and that $\mathcal{E}$ separates the edges of $E(C) \cap E(H)$ among themselves and from all other edges of G. Moreover, $\mathcal{D}$ separates the edges of $E(H) \setminus E(C)$ from all other edges. The final component of $\mathcal{C}$ will separate the edges of $E(H) \setminus E(C)$ from one another.

Note that every edge in $E(H) \setminus E(C)$ has both endvertices in $V(H) = S \cup N(S)$. Let $v_1, \ldots, v_h$ denote the vertices in $V(H)$, labeled following the cyclic order in which they appear in C. From this point onward, whenever we refer to an edge v_iv_j, we will always assume that $i < j$. We say that edges v_iv_j and v_rv_s *cross* each other if either $i < r < j < s$ or $r < i < s < j$. For given integers k and ℓ, consider the two matchings

$$M_k = \{v_iv_j \in E(H) \setminus E(C) : j - i = k\}$$
$$N_\ell = \{v_iv_j \in E(H) \setminus E(C) : j - 2i = \ell\}$$

Note that at most $3h \leqslant 9|S|$ of these matchings are nonempty, because M_k is empty whenever $k < 2$ or $k > h-1$, and N_ℓ is empty if $\ell < -h+2$ or $\ell > h-2$. We claim that the nonempty matchings separate the edges in $E(H) \setminus E(C)$. Pick two edges v_iv_j and v_rv_s. If $j - i \neq s - r$, then M_{j-i} separates v_iv_j from v_rv_s and, moreover, M_{s-r} separates v_rv_s from v_iv_j. Similarly, if $j - 2i \neq s - 2r$, then N_{j-2i} separates v_iv_j from v_rv_s and N_{s-2r} separates v_rv_s from v_iv_j. Finally, it is easy to check that $j-i = s-r$ and $i - 2j = s - 2r$ if and only if $i = r$ and $j = s$, that is, if and only if $v_iv_j = v_rv_s$. We conclude that every pair of distinct edges in $E(H) \setminus E(C)$ is separated by these matchings.

To construct $\mathcal{H}$ we shall cover each nonempty M_k (respectively, N_ℓ) using at most 4 cycles in $M_k \cup C$ (respectively, $N_\ell \cup C$) each. A trivial yet crucial observation we shall use here is that if $M \subseteq M_k$ (or $M \subseteq N_\ell$) is a set of pairwise crossing edges and $|M|$ is odd, then $M \cup C$ contains a cycle which covers M (see Figure 2). More generally, if M admits a partition $\bigcup_\alpha S_\alpha^{(M)}$ such that

(i) each $S_\alpha^{(M)}$ is formed by odd number of pairwise crossing edges, and

(ii) each pair of distinct edges $v_iv_j \in S_\alpha^{(M)}$ and $v_rv_s \in S_\beta^{(M)}$ cross if and only if $\alpha = \beta$,

then $M \cup C$ contains a cycle which covers M (see Figure 2).

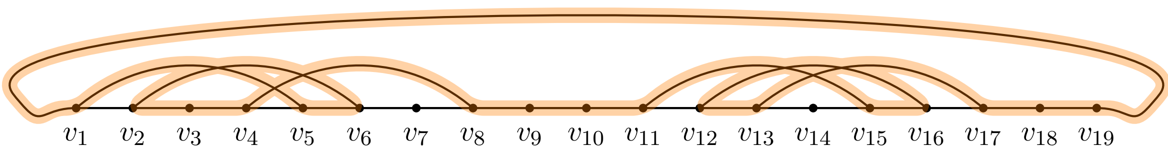

Figure 2: A cycle covering an odd number of pairwise crossing edges (v_2v_9, v_4v_{11} and v_8v_{15}).

Indeed, it turns out that each of the matchings M_k and N_ℓ admits a 4-piece partition such that each part satisfies both *(i)* and *(ii)*. Consequently, each nonempty M_k and each N_ℓ can be covered by at most four cycles using only edges in the matching and in C. The required partitions will be obtained by splitting each matching into two, twice. Given positive integers u, let $f(u)$ be the largest integer such that $2^q(\ell+1)-\ell \leqslant u$. For all k, all ℓ and all $\pi \in \{0,1\}$, put

$$M_{k,\pi} = \{v_iv_j \in E(H) \setminus E(C) : \lfloor i/k \rfloor \equiv \pi \pmod 2\}$$
$$N_{\ell,\pi} = \{v_iv_j \in E(H) \setminus E(C) : \quad f(i) \equiv \pi \pmod 2\}.$$

Let M be an arbitrary $M_{k,\pi}$ or $N_{\ell,\pi}$. We claim that M admits a partition $\bigcup_\alpha S_\alpha^{(M)}$ such that distinct edges of M cross if and only if they belong to the same part.

Proof of claim (partition of $M_{k,\pi}$). We begin with the case $M = M_{k,\pi}$. Let v_iv_j and v_rv_s be distinct edges in $M_{k,\pi}$. Without loss of generality, we assume $i < r$. By definition, $j = i+k$ and $s = r+k$. Since $\lfloor i/k \rfloor$ and $\lfloor r/k \rfloor$ have the same parity, then either $\lfloor i/k \rfloor = \lfloor r/k \rfloor$ or $\lfloor r/k \rfloor - \lfloor i/k \rfloor \geqslant 2$. In the former case, we have that

$$r < \left(\left\lfloor \frac{r}{k} \right\rfloor + 1\right) k = \left(\left\lfloor \frac{i}{k} \right\rfloor + 1\right) k = i + k = j,$$

so v_iv_j and v_rv_s cross, while in the latter they do not, since

$$j = i + k < \big(\lfloor i/k \rfloor + 1\big)k + k \leqslant \big(\lfloor r/k \rfloor - 1\big)k + k \leqslant r.$$

Hence the crossing relation defines equivalence classes among the edges in $M_{k,\pi}$, and thus the a partition of M satisfying *(ii)* exists.

Proof of claim (partition of $N_{\ell,\pi}$). The case $M = N_{\ell,\pi}$ is similar. Consider distinct $v_{i'}v_{j'}$ and $v_{r'}v_{s'}$ in $N_{\ell,\pi}$, where without loss of generality we assume $i' < r'$. By definition, either $f(i') = f(r')$ or $f(r') - f(i') \geqslant 2$. In the former case $v_{i'}v_{j'}$ and $v_{r'}v_{s'}$ must cross, since

$$r' \leqslant 2^{f(r')+1}(\ell+1) - \ell = 2^{f(i')+1}(\ell+1) - \ell \leqslant 2\big(2^{f(i')}(\ell+1) - \ell\big) + \ell \leqslant 2i' + \ell = j'.$$

On the other hand, if $f(r') - f(i') \geqslant 2$, then

$$j' = 2i' + \ell \leqslant 2\big(2^{f(i')+1}(\ell+1) - \ell\big) + \ell \leqslant 2^{f(i')+2}(\ell+1) - \ell \leqslant 2^{f(r')+2}(\ell+1) - \ell \leqslant r',$$

and consequently $v_{i'}v_{j'}$ and $v_{r'}v_{s'}$ do not cross. As before we conclude that M admits a partition $\bigcup_\alpha S_\alpha^{(M)}$ which satisfies *(ii)*.

Returning to the proof of the theorem, we complete our partitioning by refining each one of the nonempty matchings $M_{k,\pi}$ and $N_{\ell,\pi}$ (for each k, ℓ and π) further into two pieces each, so that any matching after the refinement also satisfies *(i)* (note that partition refinement does not break *(ii)*). More precisely, by *(ii)*, $M_{k,\pi}$ has a natural partition $\bigcup_\alpha S_\alpha^{(M_{k,\pi})}$ into equivalence classes such that edges in the same class are pairwise crossing and edges in distinct classes do not cross. Form $M_{k,\pi}^1$ by selecting arbitrarily one edge from each even-sized equivalence class, and let $M_{k,\pi}^2 = M_{k,\pi} \setminus M_{k,\pi}^1$ be the remaining edges of M_k (i.e., $M_{k,\pi}^1$ contains at most one edge from each equivalence class, and $M_{k,\pi}^2$ contains an odd number of edges from each equivalence class). We use the same criterion for partitioning $N_{\ell,\pi}$ into $N_{\ell,\pi}^1 \cup N_{\ell,\pi}^2$.

Note that each part resulting from this refinement satisfies both *(i)* and *(ii)*. It follows that there exists a collection $\mathcal{H}$ of at most $4 \cdot 9|S| = 36|S|$ cycles such that each nonempty M_k (respectively, N_ℓ) is covered by a at most 4 cycles in $M_k \cup E(C)$ (respectively, $N_\ell \cup E(C)$), as desired.

Note that $\mathcal{H}$ separates the edges in $E(H) \setminus E(C)$ from $E(G)$, and contains at most $36|S|$ elements. Since $|\mathcal{E}| \leqslant 2|S|$ and $|\mathcal{D}| \leqslant 3|S|$, we have that $\mathcal{C} = \mathcal{D} \cup \mathcal{E} \cup \mathcal{H}$ has at most $41|S|$ edges and paths. Hence, $\mathcal{C}' \cup \mathcal{C}$ is a cycle separating system of G with cardinality at most $41(n-|S|) + 41|S| = 41n$ as desired. This completes the proof. □

3 Concluding remarks

In this article, we have shown that every n-vertex graph admits a separating system consisting of $\mathrm{O}(n)$ edges and cycles (which we call cycle separating systems for short). This is, in at least two ways, a natural extension of previous results about the existence of path separating systems. On the one hand, a cycle separating system immediately yields a path separating system (obtained by breaking each cycle into two paths). On the hand, since paths and cycles are, respectively, subdivisions of K_2 and K_3, the following question immediately suggests itself.

Question 6. Is it true that for every natural $t \geqslant 2$, every n-vertex graph admits a separating system consisting of $\mathrm{O}_t(n)$ edges and subdivisions of K_t?

Note that edges are necessary in the separating systems in Question 6, because a union of disjoint K_{t-1} cliques has linearly many edges and no K_t subdivision. This follows in more generality from a classical result of Mader, stating that for every t there exists $f(t)$ such that every graph free of a K_t subdivision has average degree at most $f(t)$.

Our Theorem 2 and the results in [2] confirm the conjecture for $t \leqslant 3$. In a forthcoming article, the authors extend this for $t = 4$ as well, but to the best of our knowledge no further cases have been settled.

References

[1] J. Balogh, B. Csaba, R. R. Martin, and A. Pluhár. On the path separation number of graphs. *Discrete Appl. Math.*, **213** (2016), 26–33.

[2] M. Bonamy, F. Botler, F. Dross, T. Naia, and J. Skokan. Separating the edges of a graph by a linear number of paths. *Advances in Combinatorics* (2023), October.

[3] S. Brandt, H. Broersma, R. Diestel, and M. Kriesell. Global connectivity and expansion: long cycles and factors in f-connected graphs. *Combinatorica*, **26** (2006), 17–36.

[4] V. Falgas-Ravry, T. Kittipassorn, D. Korándi, S. Letzter, and B. P. Narayanan. Separating path systems. *J. Comb.*, **5** (2014), 335–354.

[5] C. G. Fernandes, G. O. Mota, and N. Sanhueza-Matamala. Separating path systems in complete graphs. In *Latin American Symposium on Theoretical Informatics, Springer* (2024), 98–113.

[6] G. Kontogeorgiou and M. Stein. An exact upper bound for the minimum size of a path system that weakly separates a clique. *arXiv preprint 2403.08210.*

[7] S. Letzter. Separating paths systems of almost linear size, 2022. *Trans. Amer. Math. Soc.* (to appear).

[8] L. Pyber. An Erdős–Gallai conjecture. *Combinatorica*, **5** (1985), 67–79.

[9] A. Rényi. On random generating elements of a finite boolean algebra. *Acta Sci. Math. Szeged*, **22** (1961), 75–81.