A covering problem for zonotopes and Coxeter permutahedra

Gyula Károlyi*[1,2]

[1]HUN–REN Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13–15, H–1053 Budapest, Hungary
[2]Department of Algebra and Number Theory, Eötvös University, Pázmány P. sétány 1/C, H–1117 Budapest, Hungary

Abstract

An almost cover of a finite set in the affine space is a collection of hyperplanes that together cover all points of the set except one. According to the Alon-Füredi theorem, every almost cover of the vertex set of an n-dimensional cube requires at least n hyperplanes. Here we investigate a possible generalization of this result to Coxeter permutahedra: convex polytopes whose vertices form the orbit of a generic point under the action of a finite reflection group.

1 Introduction

An almost cover of a finite set in the affine space is a collection of hyperplanes that together cover all points of the set except one. According to a classical result of Jamison [11], an almost cover of the n-dimensional affine space over the q-element finite field requires at least $(q-1)n$ hyperplanes. Equivalently, to pierce every affine hyperplane in $\mathbb{F}_q^n$ one needs at least $(q-1)n+1$ points, see [5]. See also [4] for further results in finite geometries. Another example is the Alon–Füredi theorem [2]: *Every almost cover of the vertex set of an n-dimensional cube requires at least n hyperplanes.*

Consider those points in the n-dimensional space whose coordinates form a permutation of the first n positive integers. The elements of this set P_n are the vertices of a convex $(n-1)$-dimensional polytope called the permutahedron (spelled also as permutohedron) Π_{n-1}. For $n=3$ it is a regular hexagon, for $n=4$ a truncated octahedron. This polytope has many fascinating properties and can be used to illustrate various concepts in geometry, combinatorics and group theory, see [13]. Our starting point is the following analogue of the Alon–Füredi theorem observed by Hegedüs, see [8].

Theorem 1. *Every almost cover of the vertices of Π_{n-1} consists of at least $\binom{n}{2}$ hyperplanes. This bound is sharp.*

A zonotope is a convex polytope that can be represented as the Minkowski sum of a finite number of line segments. A collection of line segments is called nondegenerate if no two of the segments are parallel to each other. Each zonotope Z can be written as the Minkowski sum of a nondegenerate collection of line segments, unique up to translations. The number of the summands, denoted by $\mathrm{rk}(Z)$, we call the rank of Z. In [8] we suggested that the above result and the Alon–Füredi theorem must be representatives in a more general famework.

Conjecture 2. *Every almost cover of the vertices of a zonotope Z consists of at least $\mathrm{rk}(Z)$ hyperplanes.*

*Email: karolyi.gyula@renyi.hu

Apart from some small examples, all zonotopes for which we were able to verify this hypothesis turned out to be Coxeter permutahedra. Our purpose here is to initiate a systematic study of the almost covers of their vertex sets based on a polynomial method colloquially referred to as the application of the Combinatorial Nullstellensatz.

We express our gratitude to Günter M. Ziegler for identifying one of our first examples as a permutahedron of type B, and to Francesco Santos for drawing the beautifully illuminating paper [6] of Fomin and Reading to our attention. For additional background information we refer to [9, 10].

2 Two elementary examples

The 2-dimensional zonotopes of rank r are exactly the centrally symmetric convex $2r$-gons, and every almost cover of such a polygon with lines requires at least r lines. There are two types of them that occur as zonotopal Coxeter permutahedra: regular $2r$-gons and equiangular $2r$-gons (r even) with alternating edge lengths. (The vertices of) any prism over such polygons have almost covers of size $r+1$. An elementary argument using a simple modular invariant reveals that r planes do not suffice.

Theorem 3. *Let Z be a prism over a regular $2n$-gon. Then every almost cover of the vertices of Z consists of at least $\mathrm{rk}(Z) = n+1$ planes.*

Theorem 4. *Let Z be a prism over an equiangular $4n$-gon having alternating edge lengths. Then every almost cover of the vertices of Z consists of at least $\mathrm{rk}(Z) = 2n+1$ planes.*

3 The polynomial toolkit

The Combinatorial Nullstellensatz, formulated by Noga Alon in the late nineties, describes, in an efficient way, the structure of multivariate polynomials whose zero-set includes a Cartesian product over a field $\mathbb{F}$. This characterization immediately implies ([1]) the first part of the following theorem.

Theorem 5. *Let $S_1, \ldots, S_n$ be subsets of $\mathbb{F}$, $|S_i| = k_i$, and let f be a polynomial in $\mathbb{F}[x] = \mathbb{F}[x_1, \ldots, x_n]$ whose degree is at most $\sum_{i=1}^{n}(k_i - 1)$.*

(i) If $f(s) = 0$ for every $s \in S_1 \times \cdots \times S_n$, then the coefficient of the monomial $\prod_{i=1}^{n} x_i^{k_i-1}$ in f is zero.

(ii) If $f(s) = 0$ for all but one element $s \in S_1 \times \cdots \times S_n$, then the coefficient of the monomial $\prod_{i=1}^{n} x_i^{k_i-1}$ in f is not zero.

The second part can be derived directly from (i) rather easily and is contained implicitly in many works, e.g. it is a very special case of Corollary 4.2 in [3]. The result has innumerable variations with even more different proofs, see e.g. [12]. Apparently they all depend on two basic principles: reduction modulo a standard Gröbner basis and Lagrange interpolation. It also implies the following immediate consequence of Theorem 5 in [2] we find particularly useful for the present work.

Theorem 6. *Let $S_1, \ldots, S_n$ be nonempty subsets of $\mathbb{F}$, $B = S_1 \times \cdots \times S_n$. If a polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ vanishes at every point of B except one, then its degree is at least $\sum_{i=1}^{n}(|S_i| - 1)$.*

For a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ set $V(f) = \{a \in \mathbb{R}^n \mid f(a) = 0\}$; it is called a hypersurface of degree $\deg f$. Note that the union of m hyperplanes is a hypersurface of degree m. Thus an almost cover of $X \subseteq \mathbb{R}^n$ is a hypersurface satifying $X \setminus \{v\} \subseteq V(f)$, $v \notin V(f)$ for some $v \in X$ and a polynomial f that splits into linear factors over $\mathbb{R}$. For an arbitrary hypersurface $V(f)$ satisfying the above two conditions for X and v we say that it is an almost cover of X: it covers every point of X except v. Throughout this work we are going to employ the following consequence of Theorem 6.

Corollary 7. *Let* $\emptyset \neq X \subseteq B = S_1 \times \cdots \times S_n \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[x_1, \ldots, x_n]$ *and* $d = (\sum_{i=1}^n |S_i|) - n - \deg f$. *If* $X = B \setminus V(f)$, *then every hypersurface which is an almost cover of* X *has degree at least* d.

For example, the Alon–Füredi theorem follows with the choice $S_i \equiv \{0,1\}$, $X = B$, $f = 1$. For the first statement in Theorem 1 one can use $S_i \equiv \{1, 2, \ldots, n\}$, $X = P_n$, $f = \prod_{1 \le i < j \le n}(x_j - x_i)$.

4 Prisms over permutahedra

Here we demonstrate how Theorem 5 can be used via a polynomial invariant to verify Conjecture 2 for prisms over permutahedra. Because of affine invariance it is enough to prove it for the prism whose bases are Π_{n-1} and $-\Pi_{n-1} = \Pi_{n-1} - (n+1)(e_1 + \cdots + e_n)$, where $e_1, \ldots, e_n$ is the standard orthonormal basis for $\mathbb{R}^n$.

Theorem 8. *Every almost cover of* $P_n \cup (-P_n)$ *consists of at least* $\binom{n}{2} + 1$ *hyperplanes.*

Proof. Let $m = \binom{n}{2}$ and suppose that the hyperplanes H_i, $1 \le i \le m$ cover every point of $P_n \cup (-P_n)$ except v. By symmetry, we may assume that $v \in -P_n$. The hyperplane H_i is defined by an equation $f_i(x) = a_i$ where f_i is a linear form. Consider the Vandermonde polynomial $V(x) = \prod_{i<j}(x_j - x_i)$. The polynomial

$$f(x) = V(x) \prod_{i=1}^{m} (f_i(x) - a_i))$$

of degree $n(n-1)$ vanishes at every point of the Cartesian product $\{1, 2, \ldots, n\}^n$. By Theorem 5 (i), the coefficient of the monomial $\prod_{i=1}^n x_i^{n-1}$ in f must be zero.

On the other hand, the polynomial f attains the value 0 at every point of the Cartesian product $\{-1, -2, \ldots, -n\}^n$ except v. That is, the polynomial

$$g(x) = f(-x) = (-1)^{\binom{n}{2}} V(x) \prod_{i=1}^{m} (-f_i(x) - a_i)) = V(x) \prod_{i=1}^{m} (f_i(x) + a_i))$$

of degree $n(n-1)$ vanishes at every point of the Cartesian product $\{1, 2, \ldots, n\}^n$ except $-v$. By Theorem 5 (ii), the coefficient of the monomial $\prod_{i=1}^n x_i^{n-1}$ in g must be nonzero. Since the degree $n(n-1)$ parts of the polynomials f and g are identical, we arrive at a contradiction. □

5 Reflection groups, root systems and Coxeter permutahedra

Let V be an n-dimensional real euclidean space with orthonormal basis $e_1, \ldots, e_n$. Here and in what follows we identify the vectors of V with the points of $\mathbb{R}^n$. For a nonzero vector $\alpha \in V$ we denote by s_α the orthogonal reflection in the linear hyperplane H_α orthogonal to α. Thus, $s_\alpha(\alpha) = -\alpha$. A finite reflection group acting on V is any finite group generated by (a nonempty set of) such reflections. A root system Φ is a set of nonzero vectors satisfying $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}$ and $s_\alpha(\Phi) = \Phi$ for every $\alpha \in \Phi$. Crytallographic root systems satisfy an extra integrality condition. The group $W(\Phi)$ (called Weyl group in the crystallographic case) of orthogonal transformations generated by the reflections s_α, $\alpha \in \Phi$ is always a finite reflection group in which the reflections exhaust Φ. Thus, Φ is invariant under the action of W. Conversely, if W is a finite reflection group, then the unit vectors α for which $s_\alpha \in W$ form a root system Φ for which $W = W(\Phi)$. If the vectors in Φ form one orbit under the action of W, then $W = W(\Phi')$ if and only if $\Phi' = c\Phi$ for some $0 \neq c \in \mathbb{R}$. On the other hand, if Φ is the union of more than one orbits, then the common length of the vectors in an orbit may be scaled arbitrarily for each orbit. Thus, if $W = \mathrm{I}_2(m)$ is the symmetry group of a regular m-gon centered at the origin, then each corresponding root system has $2m$ elements, which form one orbit if m is odd and splits into two orbits of equal size if m is even.

Let $W = W(\Phi)$ be a finite reflection group. For any point $a \in \mathbb{R}^n$, consider its orbit $W(a)$. The point a is called *generic* with respect to W, if $|W(a)| = |W|$, or equivalently, $a \notin \bigcup_{\alpha\in\Phi} H_\alpha$. In this case $W(a)$ is the vertex set of a (not necessarily full dimensional) convex polytope $\Pi W(a)$, referred to as a *W-permutahedron*, or a Coxeter permutohedron of type W. Thus, a permutahedron of type $\mathrm{I}_2(m)$ is either a regular $2m$-gon, or an equiangular $2m$-gon with alternating edge lengths (the latter being a zonotope only for m even), and each such polygon centered at the origin can be obtained as a Coxeter permutahedron for an appropriate choice of Φ. All vertices except one can be covered by m, but not less lines.

A root system Φ is irreducible if it cannot be partitioned into two subsets lying in two nontrivial orthogonal complements of V, or equivalently, if $W(\Phi)$ is not the direct sum of two proper subgroups acting as reflection groups on two such subspaces. Theorems 3 and 4 thus read as follows: *Every almost cover of a zonotopal permutahedron of type $\mathrm{I}_2(m) \oplus \mathrm{A}_1$ requires at least $m+1$ hyperplanes.* Note that the group contains exactly $m+1$ reflections.

Next consider the reflection group A_{n-1} acting on $\mathbb{R}^n$, generated by the reflections in the hyperplanes of equation $x_{i+1} = x_i$, $i = 1, \dots, n-1$. It is isomorphic to the symmetric group S_n, and a point is generic if and only if all its coordinates are different. Thus we have $\Pi_{n-1} = \Pi\mathrm{A}_{n-1}(1, 2, \dots, n)$, and Thm 1 coupled with the remark following its proof in [8] can be read as follows: *Every almost cover of the vertices of a Coxeter permutahedron of type A_{n-1} consists of at least $\binom{n}{2}$ hyperplanes.* The bound is also sharp. Note that the vectors $e_i - e_j$ ($i \neq j$) form a root system for A_{n-1}, so the bound equals the number of reflections contained in A_{n-1}. In general, for a reflection group $W = W(\Phi)$, the number of reflections contained in W is $N(W) = |\Phi|/2$.

It is not difficult to prove an analogue of Thm 1 for permutahedra of type B. The hyperoctahedral group B_n acting on $\mathbb{R}^n$ is generated by the reflections in the hyperplanes of equation $x_{i+1} = x_i$, $i = 1, \dots, n-1$, together with the reflection in the hyperplane $x_1 = 0$; it contains A_{n-1} as a subgroup. Altogether it contains n^2 reflections in the hyperplanes $x_i = \varepsilon x_j$ ($1 \leq i < j \leq n$, $\varepsilon = \pm 1$) and $x_i = 0$ ($1 \leq i \leq n$). Thus, $N(\mathrm{B}_n) = n^2$. A point $a = (a_1, \dots, a_n)$ is generic if and only if $a_i \neq 0$ for all i and $|a_i| \neq |a_j|$ for all $i \neq j$. Thus every orbit of a generic point is of the form

$$B_n(a) = \{\varepsilon_1 a_{\pi(1)} + \cdots + \varepsilon_n a_{\pi(n)} \mid \varepsilon_i = \pm 1, \pi \in S_n\}$$

for some $a \in \mathbb{R}^n$ with coordinates $0 < a_1 < \cdots < a_n$.

Theorem 9. *Every almost cover of the vertices of a Coxeter permutahedron of type B_n consists of at least n^2 hyperplanes. This bound is sharp.*

Proof. The vertex set of the permutahedron $\Pi\mathrm{B}_n(a)$ with $0 < a_1 < \cdots < a_n$ is contained in the Cartesian product $S_1 \times \cdots \times S_n$ where $S_i = \{a_i, -a_i \mid 1 \leq i \leq n\}$, and each point in $(S_1 \times \cdots \times S_n) \setminus \mathrm{B}_n(a)$ is a root of the polynomial

$$f(x) = \prod_{1 \leq i < j \leq n} (x_j - x_i)(x_j + x_i)$$

of degree $n(n-1)$. According to Corollary 7, every almost cover of $\mathrm{B}_n(a)$ consists of at least

$$\left(\sum_{i=1}^{n} |S_i|\right) - n - \deg f = 2n^2 - n - n(n-1) = n^2$$

hyperplanes. To see that the bound cannot be improved, notice that the hyperplanes $x_i = a_j$ ($i < j$), $x_i = -a_j$ ($i \leq j$) cover every vertex but $a = (a_1, a_2, \dots, a_n)$. □

The study of almost covers of the vertices of permutahedra of type D is more subtle. The group D_n is the subgroup of index 2 in B_n generated by the reflections in the hyperplanes of equation $x_{i+1} = x_i$, $i = 1, \dots, n-1$, together with the reflection in the hyperplane $x_2 = -x_1$. Altogether it contains

$n(n-1)$ reflections in the hyperplanes $x_i = \varepsilon x_j$ ($1 \le i < j \le n$, $\varepsilon = \pm 1$). A point $a = (a_1, \dots, a_n)$ is generic if and only if $|a_i| \ne |a_j|$ for all $i \ne j$. Thus every orbit of a generic point is of the form

$$\mathrm{D}_n(a) = \{\varepsilon_1 a_{\pi(1)} + \cdots + \varepsilon_n a_{\pi(n)} \mid \pi \in S_n, \varepsilon \in E\}$$

for some $a \in \mathbb{R}^n$ with coordinates $-a_2 < a_1 < a_2 < \cdots < a_n$, where E is either of the two subsets of $\{-1, 1\}^n$ that consists of all vectors in which the number of -1 coordinates are the same modulo 2.

Theorem 10. *Every almost cover of the vertices of a Coxeter permutahedron of type* D_n *consists of at least* $n(n-1)$ *hyperplanes. This bound is sharp in the following sense: if* a *is a generic point one of whose coordinates is 0, then* $\mathrm{D}_n(a)$ *has an almost cover of size* $n(n-1)$.

Proof. It is very similar to the previous one if the vertices of the permutahedron have a 0 coordinate. Otherwise we may assume by symmetry that the vertex set is $\mathrm{D}_n(a)$ with $0 < a_1 < \cdots < a_n$. In this case we can apply Corollary 7 with the polynomial

$$f(x) = \prod_{1 \le i < j \le n} (x_j - x_i)(x_j + x_i) \left(\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} a_i \right)$$

of degree n^2. □

These results suggest that the following might be true.

Conjecture 11. *For a finite reflection group* W*, every almost cover of the vertices of a permutahedron of type* W *consists of at least* $N(W)$ *hyperplanes.*

In contrast, all vertices of a Coxeter permutahedron are contained in a single hypersurface of degree 2, namely a sphere centered at the origin.

6 Zonotopal permutahedra

For the reflection group $W = \mathrm{A}_n$, the orbit of any generic point contains a unique point $a = (a_1, \dots, a_{n+1})$ with $a_1 < \cdots < a_{n+1}$. Similarly, for $W = \mathrm{B}_n$, the orbit of any generic point contains a unique point $a = (a_1, \dots, a_n)$ with $0 < a_1 < \cdots < a_n$. For such points it is known that the Coxeter permutahedron $\Pi W(a)$ is a zonotope if and only if the coordinates a_i form an arithmetic progression, see [7, Thm 4.13]. We can prove an analogous statement for permutahedra of type D, and in fact all these results can be viewed as special cases of a more general phenomenon. For a root system Φ, consider any set Φ^+ of positive roots. The Minkowski sum of the line segments $[-\alpha/2, \alpha/2]$, $\alpha \in \Phi^+$, independent of the choice of Φ^+ we denote by $Z(\Phi)$. Then $\mathrm{rk}(Z(\Phi)) = N(W(\Phi))$.

Theorem 12. *Let* W *be a finite reflection group with a corresponding root system* Φ*. Then* $Z(\Phi)$ *is a permutahedron of type* W*.*

The reflection group W is called essential if it acts on V without nonzero fixed points. In general, $V = U \oplus U'$, where W is essential relative to U and the orthogonal complement U' consists of all fixed points of W.

Theorem 13. *A permutahedron* Π *of type* W *is a zonotope if and only if there exists a root system* Φ *with* $W(\Phi) = W$ *and a vector* $u \in U'$ *such that* $\Pi = Z(\Phi) + u$.

Although it is not likely that Conjectures 2 and 11 for $Z(\Phi)$ in general can be attacked by our methods, it is possible to say something more for crystallographic root systems. We call a zonotope $Z \subset \mathbb{R}^n$ *special* if there exist finite sets $S_1, \dots, S_n \subset \mathbb{R}$ and a polynomial f such that the vertex set X of Z is $(S_1 \times \cdots \times S_n) \setminus V(f)$ and

$$\mathrm{rk}(Z) \le |S_1| + \cdots + |S_n| - n - \deg f.$$

According to Corollary 7, every almost cover of the vertices of a special zonotope Z consists of at least $\mathrm{rk}(Z)$ hyperplanes. Now for an irreducible crystallographic root system Φ, $Z(\Phi)$ is special if the type of Φ is $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$ or G_2. Moreover, if V is the sum of the orthogonal subspaces V_1, V_2 and $\Phi = \Phi_1 \cup \Phi_2$ with $\Phi_i = \Phi \cap V_i$, then $Z(\Phi)$ is the product polytope $Z(\Phi_1) \times Z(\Phi_2)$. In general, $\mathrm{rk}(Z_1 \times Z_2) = \mathrm{rk}(Z_1) + \mathrm{rk}(Z_2)$ holds for arbitrary zonotopes Z_1, Z_2. Then the following construction yields further examples for which these conjectures hold.

Theorem 14. *If $Z_1, \ldots, Z_k$ are special zonotopes, then so is $Z_1 \times \cdots \times Z_k$.*

For the crystallographic root system Φ of type F_4, the vertex set of $Z(\Phi)$ splits into three B_4-orbits. We can construct an almost cover of size $24 = \mathrm{rk}(Z(\Phi))$, but we do not see if our method suits a proof that this is best possible.

7 Conclusion

We investigated how the polynomial method can be used to study almost covers of vertex sets of zopotopes and Coxeter permutahedra. In the meantime, Conjecture 2 was refuted by Gábor Damásdi, whereas Conjecture 11 was verified by Péter Frenkel.

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