Characterization of the equality in some discrete isoperimetric and Brunn-Minkowski type inequalities*

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Abstract

Lattice cubes are optimal sets with respect to several recent inequalities in Discrete Geometry, including analogues - for the cardinality - of both the Brunn-Minkowski inequality and the isoperimetric inequality. While a general characterization of the equality case has not been obtained thus far, we show that when the cardinality is suitable lattice cubes do comprise the unique optimal sets with respect to the aforementioned discrete inequalities, and discuss the underlying techniques of the proof.

1 Introduction

The isoperimetric inequality is one of the most classical results in Geometry, dating back to antiquity in the case of the planar version. Informally, it asserts that Euclidean balls minimize the ratio between the surface area and the volume. More rigorously, given a bounded set with non-empty interior $M \subset \mathbb{R}^n$ and the unit Euclidean ball of dimension n, B_n, one has

$$\frac{\mathcal{S}(M)^n}{\operatorname{vol}(M)^{n-1}} \geq \frac{S(B_n)^n}{\operatorname{vol}(B_n)^{n-1}}, \tag{1}$$

where $\operatorname{vol}(\cdot)$ is the volume (Lebesgue measure) and $\mathcal{S}(\cdot)$ is the surface area measure. Equivalently, since $\mathcal{S}(B_n) = n \operatorname{vol}(B_n)$, one has

$$\mathcal{S}(M) \geq n \operatorname{vol}(M)^{1-\frac{1}{n}} \operatorname{vol}(B_n)^{\frac{1}{n}}.$$

If one restricts the inequality to convex sets only, then Euclidean balls characterize inequality (1).

On the other hand, the classic Brunn-Minkowski inequality provides a relationship between the volume and the addition of sets. Namely, given two non-empty compact sets $K, L \subset \mathbb{R}^n$, it states that

$$\operatorname{vol}(K+L)^{1/n} \geq \operatorname{vol}(K)^{1/n} + \operatorname{vol}(L)^{1/n}, \tag{2}$$

with equality if and only if K and L are either homothetic or lie in parallel hyperplanes. Here, the addition being employed is the standard pointwise addition - or *Minkowski addition* - of sets.

In other words, the functional $\operatorname{vol}(\cdot)^{1/n}$ is concave in the family of non-empty compact sets. In fact, the result holds true even when K, L and $K + L$ are just Lebesgue measurable.

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The Brunn-Minkowski inequality gave way to the development of a rich theory of related results and generalizations to other contexts, such as functional analogues (e.g. [6, 7, 21, 22]); extensions for other measures, like the mixed volumes (see [1]); a reverse form [20]; or applications to the concentration of measure (see, e.g., [2]), among others. Notably, the isoperimetric inequality follows easily from the Brunn-Minkowski inequality (one may also derive the isodiametric inequality, and even its stronger counterpart for the mean width, Urysohn's inequality). For a thorough treatment of the Brunn-Minkowski inequality we refer to the comprehensive surveys [3, 9] and the excellent monographs [12, 26].

In recent decades, there has been considerable effort translating some of these results, and many others in the context of Convex Geometry, to the discrete setting. For the isoperimetric inequality, see e.g. [4, 5, 8, 16, 23, 28]; for the Brunn-Minkowski inequality, see e.g. [10, 11, 13, 14, 16, 17, 18, 19, 24, 25, 27]. Most commonly, one either considers finite subsets of lattices equipped with the cardinality measure, or regular bounded sets equipped with the lattice point enumerator measure. In this work we will focus on the former approach.

In the next section we will highlight a few of the inequalities discussed above, for which the announced characterizations have been obtained. In the final section we will provide a general outlook of the proof and its underlying ideas.

2 Preliminaries and overview

In [23], building on previous ideas from [5, 28], the authors obtained an isoperimetric inequality for the cardinality in the setting of $\mathbb{Z}^n$ and $\mathbb{N}^n$. In order to introduce it, we first need to recall their notion of boundary of a set in this context (see also Figure 1)

Definition 1. *The boundary of a discrete set* $A \subset \mathbb{Z}^n$ *is* $\partial(A) = (A + \{-1, 0, 1\}^n) \setminus A$.

Note that this setting can be interpreted as considering the lattice endowed with the L_∞ norm. The isoperimetric problem for the cardinality can then be reformulated as finding the sets with a given fixed cardinality which minimize $|\partial(\cdot)|$. By construction, this is equivalent to simply finding the minimizers for the functional $|A + \{-1, 0, 1\}^n|$.

In order to describe said minimizers, the authors defined a complete order in $\mathbb{Z}^n$. They then showed that the *initial segments* $\mathcal{I}_r \subset \mathbb{Z}^n$ (i.e., the first r points in the order, $r \in \mathbb{N}$; see Figure 2) constitute an infinite family of minimizers, one for each cardinality.

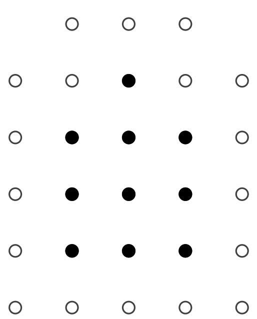

Figure 1: Boundary of a discrete set.

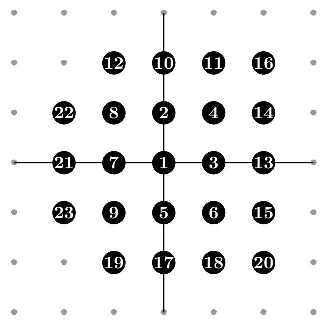

Figure 2: The set $\mathcal{I}_{23}$ with the ordering specified.

Observe that, due to them being initial segments, they are additionally all nested. We will not delve into the details of this order here, for which we instead refer to [23], as well as to [5, 10, 28], which use a similar approach with an order known as the *simplicial order*. As an example, the set from Figure 1 is $\mathcal{I}_{10}$, and the origin is $\mathcal{I}_1$. The authors obtained the following result:

Theorem 2. *[23, Theorem 1] Let* $A \subset \mathbb{Z}^n$ *with* $r = |A| > 0$. *Then*

$$\big|A + \{-1, 0, 1\}^n\big| \geq \big|\mathcal{I}_r + \{-1, 0, 1\}^n\big|. \tag{3}$$

They also considered the restriction of the order to $\mathbb{N}^n$, which gives rise to the corresponding initial segments $\mathcal{J}_r \subset \mathbb{N}^n$ of cardinality r, and derived the following analogous result:

Theorem 3. *[23, Corollary 1] Let $A \subset \mathbb{N}^n$ with $r = |A| > 0$. Then*

$$\left|\left(A + \{-1,0,1\}^n\right) \cap \mathbb{N}^n\right| \geq \left|\left(\mathcal{J}_r + \{-1,0,1\}^n\right) \cap \mathbb{N}^n\right|. \tag{4}$$

The authors noted, however, that these sets do not characterize the equality. Indeed, in the set from Figure 1, one may translate the outermost point one unit to the left or to the right, and the cardinality of the boundary will remain constant, despite these new sets not being initial segments.

It is relevant to mention now that both $\mathcal{I}_{\rho^n}$ and $\mathcal{J}_{\rho^n}$, for $\rho \in \mathbb{N}, \rho > 0$, are lattice cubes with side length ρ. Therefore, all lattice cubes are minimizers of this problem. Moreover, due to the nestedness, it follows that any initial segment is contained between two consecutive lattice cubes, that is, it is composed by a lattice cube and some additional points in its outermost layer (again, cf. Figure 1). As before, it is easy to find additional minimizers (which are not initial segments) by shifting points in this outermost layer. Even though other configurations do exist, this nevertheless suggests to study the special case of lattice cubes.

We show that, indeed, lattice cubes characterize the equality on both Theorem 2 and Theorem 3, whenever the cardinality involved is suitable:

Theorem 4. *[15, Theorem 1] Let $A \subset \mathbb{Z}^n$ with $|A| = (\rho+1)^n$ for some $\rho \in \mathbb{N}$. Then equality holds in* (3) *if and only if A is a lattice cube.*

Theorem 5. *[15, Theorem 2] Let $A \subset \mathbb{N}^n$ with $|A| = (\rho+1)^n$ for some $\rho \in \mathbb{N}$. Then equality holds in* (4) *if and only if $A = \{0,\dots,\rho\}^n$.*

It can be shown that, in fact, both settings ($\mathbb{Z}^n$ and $\mathbb{N}^n$) are equivalent, and therefore, it suffices to focus on one of them. The above results both follow from the more general result below:

Theorem 6. *[15, Theorem 17] Let $A \subset \mathbb{N}^n$ with $|A| = (\rho+1)^n$ for some $\rho \in \mathbb{N}$ and let $s \in \mathbb{N}, s > 0$. If*

$$\left|A + \{0,\dots,s\}^n\right| = \left|\mathcal{J}_{(\rho+1)^n} + \{0,\dots,s\}^n\right|,$$

then A is a lattice cube.

Though we will not delve into it, the above result can also be applied in the context of the lattice point enumerator. In particular, it implies a characterization of the equality case in a discrete isoperimetric inequality for the lattice point enumerator recently obtained (see [15, Theorem 3]).

As for the Brunn-Minkowski inequality, we highlight a discrete analogue for the cardinality obtained in [18]. The authors proved the following inequality:

Theorem 7. *[18, Theorem 3.2] For every non-empty finite sets $A, B \subset \mathbb{Z}^n$,*

$$\left|A + B + \{0,1\}^n\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n}. \tag{5}$$

The inequality is sharp (e.g., for lattice cubes), and in fact, the authors showed that it is equivalent to the classic version for the volume (2).

One may wonder whether lattice cubes also characterize the above inequality when the cardinalities are suitable. While this is not true (see Figure 3), if we also know that one of the sets is a lattice cube, then the equality does imply that the other set must be a lattice cube as well. More specifically, as a consequence of Theorem 6, we obtain the following characterization of Theorem 7:

Corollary 8. *[15, Corollary 35] Let $A \subset \mathbb{Z}^n$ be a finite set with $|A| = (\rho+1)^n$ for some $\rho \in \mathbb{N}$ and let B be a lattice cube. Then*

$$\left|A + B + \{0,1\}^n\right|^{1/n} = |A|^{1/n} + |B|^{1/n}$$

if and only if A is a lattice cube.

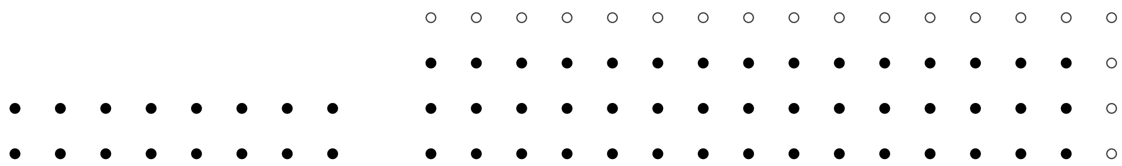

Figure 3: The set $A = \{0, \dots, 7\} \times \{0, 1\} \subset \mathbb{Z}^2$ (left) satisfies $|A| = 16$, and $A + A + \{0, 1\}^2$ (right) satisfies the equality in (5): $|A + A + \{0, 1\}^2|^{1/2} = 8 = 2|A|^{1/2}$.

3 Sketch of the proof

The overall idea is a refinement of the approach used in [23] to prove Theorems 2 and 3. As discussed, we can restrict our attention to $\mathbb{N}^n$, since the results follow for $\mathbb{Z}^n$. First, we define a transformation on $\mathbb{N}^n$ we shall refer to as *normalization.*

To normalize a given set $A \subset \mathbb{N}^n$ in the direction of the canonical vector e_i, $i = 1, \dots, n$, we consider the non-empty $(n-1)$-dimensional sections of A orthogonal to e_i and we perform the following operations (see Figure 4), which we proceed to describe informally:

1. Replace each section by the $(n-1)$-dimensional initial segment of the same cardinality.

2. Rearrange the sections in decreasing order of cardinality, such that the largest section is at the origin.

3. Translate points from the upper sections to the lower ones in a suitable way that preserves the first two properties, that is, the sections will remain initial segments ordered by cardinality.

The mathematical details missing in the above description, specially those pertaining to the third and final step, are rather technical, and therefore, we instead refer the reader to [15, Definition 28] for the precise and rigorous description.

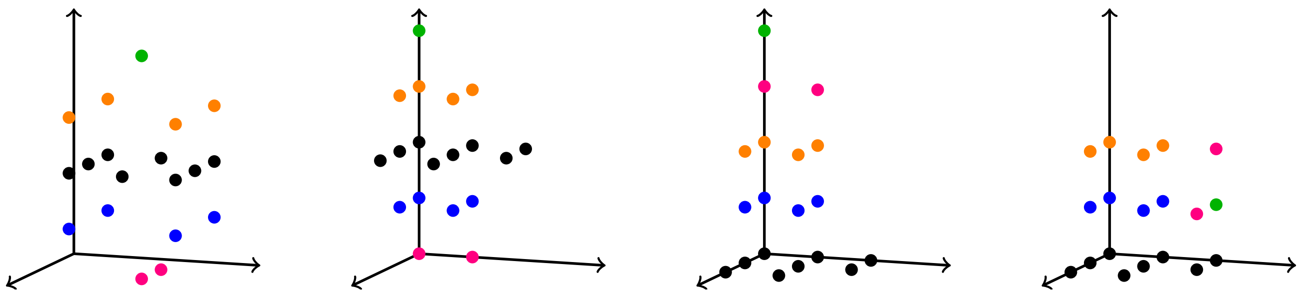

Figure 4: From left to right: a finite set, together with the same set after each step of the normalization is applied.

We will say that a set is normalized with respect to e_i if it coincides with its normalization in said direction. If one normalizes a set with respect to e_i and then with respect to e_j, for some $i, j = 1, \dots, n$, $i \neq j$, it is not guaranteed that the result will remain normalized with respect to e_i. However, it can be proved that the process is finite, in the sense that by repeatedly normalizing a set with respect to all (canonical) directions one eventually reaches a set which is normalized with respect to e_i for all $i = 1, \dots, n$ simultaneously (see [15, Lemma 30]).

Definition 9. *A set $A \subset \mathbb{N}^n$ is stable if it is normalized with respect to e_i for all $i = 1, \dots, n$ simultaneously.*

Moreover, we show that the normalization process does not increase the cardinality of the boundary (see [15, Lemma 31]), and therefore, in particular, the normalization of a minimizer is still a minimizer.

The approach then consists on reducing the study to stable minimizers, since any arbitrary minimizer can be transformed into a stable one via a finite number of normalizations.

We prove that any stable minimizer is a lattice cube (see [15, Lemma 34]). Finally, and crucially, in the proof of the main theorem it is shown that any minimizer which is normalized into a lattice cube must be a lattice cube itself. Putting everything together yields Theorem 6, and thus Theorems 4 and 5. We finish with two additional observations regarding the proofs of the results stated above.

First, the proof of the fact that any minimizer which normalizes into a lattice cube must be a lattice cube itself relies on induction on the dimension, which is possible since the $(n-1)$-dimensional sections of a minimizer are minimizers in $\mathbb{N}^{n-1}$ (see [15, Corollary 26]).

Second, the normalization process is so restrictive that a very explicit and precise description for stable sets can be obtained. The proof of the fact that stable minimizers are lattice cubes heavily depends on this description, which allows to perform direct computations on the stable set.

Namely, any stable set can be decomposed as the disjoint union of a lattice box and two additional, $(n-1)$-dimensional sets, as described in the following lemma (see also Figure 5).

Lemma 10. *Let $n \geq 3$, $\rho \geq 1$ and let $X \subset \mathbb{N}^n$ be a non-empty finite set with $|X| = (\rho+1)^n$. If X is stable, then there exist $A, B \subset \mathbb{N}^n$ such that*

$$A \subset \{0, \ldots, \rho-1\}^{n-1} \times \{\rho+1\}, \qquad \emptyset \neq B \subset \{\rho\} \times \{0, \ldots, \rho\}^{n-1}$$

and

$$X = A \cup B \cup \left(\{0, \ldots, \rho-1\} \times \{0, \ldots, \rho\}^{n-1}\right).$$

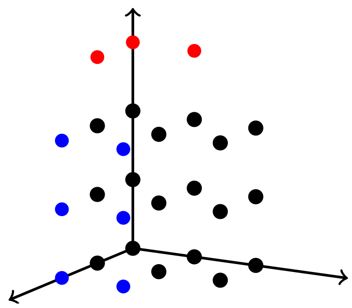

Figure 5: A stable set $X \subset \mathbb{N}^3$, highlighting the three subsets from the decomposition of Lemma 10.

Regarding the discrete Brunn-Minkowski analogue for the cardinality, we finish by obtaining Corollary 8 as a quick consequence of Theorem 6.

Proof of Corollary 8. Assume that $\left|A + B + \{0,1\}^n\right|^{1/n} = |A|^{1/n} + |B|^{1/n}$, and let $B = \{0, \ldots, s\}^n$ for some $s \in \mathbb{N}$. Then, by the minimality of initial segments (see also [15, Corollary 11]), we have

$$(\rho+s+2)^n = |A + B + \{0,1\}^n| \geq |\mathcal{J}_{(\rho+1)^n} + B + \{0,1\}^n| = (\rho+s+2)^n.$$

Thus, $|A + \{0, \ldots, s+1\}^n| = |\mathcal{J}_{(\rho+1)^n} + \{0, \ldots, s+1\}^n|$, and Theorem 6 implies A must be a lattice cube, as desired. □

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