Bounding the balanced upper chromatic number *

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Abstract

In this paper, we provide a general upper bound on the *balanced upper chromatic number of any linear hypergraph*, that is, the largest size of a vertex coloring of any linear hypergraph in which all color-class sizes differ by at most one (balanced) and each hyperedge contains at least two vertices of the same color (rainbow-free). We are particularly interested in understanding this parameter for the n-dimensional cube on t elements due to its close connection to the unexistence of a rainbow Ramsey version of the Hales-Jewett Theorem. We improve the lower and upper bounds for this hypergraph and (except for four cubes) completely determine this parameter in dimensions 2 and 3.

1 Introduction

Many classical Ramsey theory results that deal with the existence of a monochromatic object have a rainbow counterpart: a theorem that guarantees the existence of certain rainbow subset (i.e. such that no color is repeated) provided that the color set is sufficiently large and that all colors are well represented. An example of this is van der Waerden's theorem, whose rainbow counterpart for three colors was studied in [8]. The novelty in [8] was to notice that rainbow structures can be forced to appear not only by letting the number of colors grow, but also by fixing the number of colors and letting all chromatic classes be large enough. This is because the more balanced the color classes are the higher the number of rainbow substructures is. For instance, in k-colorings of the set of vertices of a t-uniform hypergraph H, the number of rainbow t-sets of vertices is higher as the coloring becomes more balanced. Therefore, when looking for a rainbow-free k-coloring of H, it is in principle harder to find one among balanced k-colorings (this, of course, depends on the structure of H). Consequently, we consider ~~here~~ *balanced colorings* of the vertices of a given hypergraph H, that is, colorings in which the cardinalities of all color classes differ in at most one. In this setting, we aim to maximize the number k of colors for which there is a balanced k-coloring of H without *rainbow hyperedges*, i.e. hyperedges where all colors appear at most once. To avoid inconsistencies with this definition, we assume that all hyperedges have size at least two.

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This maximum value, defined originally in [3], is called the *balanced upper chromatic number of H*, and it is denoted by $\overline{\chi}_b(H)$. Clearly, this parameter is related to the *upper chromatic number* $\overline{\chi}(H)$ defined as the maximum number of colors in a coloring of H (not necessarily balanced) without rainbow hyperedges, which has been the subject of study in many papers, see for instance [4, 5, 10]. Observe that, if $E(H)$ is the set of hyperedges and n is the order of H, then $\min\{|e| : e \in E(H)\} - 1 \leq \overline{\chi}_b(H) \leq \overline{\chi}(H) \leq n$. Often, lower bounds for $\overline{\chi}(H)$ are obtained by colorings with one very large color class and all other classes of size one. Evidently, such colorings do not provide lower bounds for $\overline{\chi}_b(H)$ which requires more involved constructions.

The upper balanced chromatic number of C_t^n, the n-dimensional cube over t-elements, is in close connection with the unexistence of a rainbow Ramsey version of the Hales-Jewett theorem, a central result in Ramsey theory that establishes the existence of monochromatic combinatorial lines in any finite coloring of C_t^n provided that n is sufficiently large, it was shown in [9] that, except for the case $(t, n) = (3, 2)$, for every $t \geq 3$ and every $n \geq 2$, there are balanced t-colorings of C_t^n without rainbow lines. Moreover, for every even $t \geq 4$ and every n, there are balanced $\left(\frac{t}{2}\right)^n$-colorings of C_t^n without rainbow lines [9]. This shows that $\overline{\chi}_b(C_t^n) \geq \left(\frac{t}{2}\right)^n$. In a recent work by the authors of this paper [2], it is proved that

$$\overline{\chi}_b(C_t^n) \leq \frac{3t^n - (t+2)^n}{2}, \tag{1}$$

for $t \geq \frac{2}{\sqrt[n]{2}-1}$, and that the bound is attained when $t \geq 4n - 2$.

In Section 2, we generalize the idea leading into Inequality (1), providing a general upper bound for the balanced upper chromatic number of any *linear hypergraph*, that is, a hypergraph where every two edges intersect in at most one vertex, Theorem 1. Moreover, we complete the upper bound for the case that $v < 2e$, showing that $\overline{\chi}_b(H) \leq \left\lceil 2(v+2e) - 4\sqrt{e^2+ev} \right\rceil - 1$. This bound comes very close to the known upper bounds for the upper balanced chromatic number of finite projective planes, which constitute a special family of linear hypergraphs, that were given in [3, 7].

In Section 3, we provide bounds for the balanced chromatic number of the n-dimensional cube. We present a general lower bound that follows from Theorem 1 and sharper bounds for specific values of t (improving those in [9]). Finally, Section 4 refines our results for the plane and the space.

2 General upper bound

We start presenting an upper bound for the balanced upper chromatic number of a linear hypergraph.

Theorem 1. *Let H be a linear hypergraph with v vertices and e hyperedges. Then*

$$\overline{\chi}_b(H) \leq \begin{cases} v - e & \text{if } v \geq 2e, \\ \left\lceil 2(v+2e) - 4\sqrt{e^2+ev} \right\rceil - 1 & \text{if } v < 2e. \end{cases}$$

Proof. Consider a balanced c-coloring of H for some integer $2 \leq c \leq v$. Then there is an integer $1 \leq k < v$ such that all color classes are of size k or possibly $k+1$. Let $c_k \geq 1$ and $c_{k+1} \geq 0$ be the number of classes of size k and $k+1$, respectively. Then $c = c_k + c_{k+1}$ and $v = kc_k + (k+1)c_{k+1} = ck + c_{k+1}$. Since $0 \leq c_{k+1} < c$, then k and c_{k+1} are the quotient and the remainder, respectively, when v is divided by c. That is, $k = \lfloor \frac{v}{c} \rfloor$ and $c_{k+1} = v - c\lfloor \frac{v}{c} \rfloor$. Let $\varepsilon = \frac{c_{k+1}}{c} = \frac{v}{c} - \lfloor \frac{v}{c} \rfloor$. Thus $c_{k+1} = \varepsilon c$ and $k = \frac{v}{c} - \varepsilon$, where $0 \leq \varepsilon < 1$. We say that a color *blocks* a hyperedge if at least two vertices of the hyperedge receive that color. So an *unblocked* edge is a rainbow edge. Note that at most $\binom{k}{2}c_k + \binom{k+1}{2}c_{k+1} = \binom{k}{2}c + kc_{k+1}$ hyperedges can be blocked by the distinct colors in the c-coloring. Hence, if

$$e > \binom{k}{2}c + kc_{k+1} = \binom{\frac{v}{c} - \varepsilon}{2}c + \left(\frac{v}{c} - \varepsilon\right)\varepsilon c = \frac{1}{2}\left(\frac{v}{c} - \varepsilon\right)(v + \varepsilon c - c) = \frac{1}{2c}(v - \varepsilon c)(v + \varepsilon c - c), \tag{2}$$

then there is at least one rainbow hyperedge.

First, assume that $v \geq 2e$ and let $c = v - e + 1$. To prove that any c-coloring of H has rainbow hyperedges, it is enough to verify Inequality (2). In this case, $v = (v-e+1)\cdot 1 + (e-1) = c\cdot 1 + (e-1)$ and the assumption $v \geq 2e$ implies $0 \leq e-1 < v-e+1 = c$. Thus $\varepsilon c = e-1$ and

$$\frac{1}{2c}\left(v-\varepsilon c\right)\left(v+c\varepsilon-c\right) = \frac{1}{2(v-e+1)}\left(v-(e-1)\right)\left(v+(e-1)-(v-e+1)\right) = e-1 < e.$$

Similarly, assume that $v < 2e$ and let $c = \left\lceil 2(v+2e) - 4\sqrt{e^2+ev}\right\rceil$. It is enough to show that Inequality (2) holds. Note that (rationalizing)

$$c > 2(v+2e) - 4\sqrt{e^2+ev} = \frac{2v^2}{v+2e+2\sqrt{(e^2+ev)}} = \frac{2v^2}{v+2e+\sqrt{(v+2e)^2-v^2}}. \tag{3}$$

Because $(v+2e)^2 - v^2 < (v+2e)^2$, it follows that $c > \frac{v^2}{v+2e}$. This implies that $e > \frac{v(v-c)}{2c}$, which is precisely Inequality (2) when $\varepsilon = 0$.

Assume now that $0 < \varepsilon < 1$. Thus $0 < \varepsilon(1-\varepsilon) \leq \frac{1}{4}$. Since $2e > v$ and $v \geq c$, we have that

$$\frac{v+2e}{2\varepsilon(1-\varepsilon)} > \frac{v}{\varepsilon(1-\varepsilon)} \geq 4v > v \geq c. \tag{4}$$

Also, $v^2 \geq 4v^2\varepsilon(1-\varepsilon)$. By Inequality (3) and rationalizing, we obtain

$$c > \frac{2v^2}{v+2e+\sqrt{(v+2e)^2-4v^2\varepsilon(1-\varepsilon)}} = \frac{v+2e-\sqrt{(v+2e)^2-4v^2\varepsilon(1-\varepsilon)}}{2\varepsilon(1-\varepsilon)}. \tag{5}$$

Inequalities (4) and (5) imply that

$$\frac{\sqrt{(v+2e)^2-4v^2\varepsilon(1-\varepsilon)}}{2\varepsilon(1-\varepsilon)} > \frac{v+2e}{2\varepsilon(1-\varepsilon)} - c > 0.$$

Multiplying by $2\varepsilon(1-\varepsilon)$ and squaring, we obtain $(v+2e)^2 - 4v^2\varepsilon(1-\varepsilon) > (v+2e-2\varepsilon(1-\varepsilon)c)^2$, which is equivalent to

$$e > \frac{1}{2c}\left(v^2 - vc + \varepsilon c^2 - \varepsilon^2 c^2\right) = \frac{1}{2c}\left(v-\varepsilon c\right)\left(v+\varepsilon c-c\right).$$

This is again Inequality (2) and thus there must be a rainbow hyperedge. □

The finite projective planes Π_q of order q have the same number of lines (of size $q+1$) and points, namely $v = e = q^2+q+1$. It has been shown that $\overline{\chi}_b(\Pi_q) \leq v/3$ [3, 7]. The upper bound above, which in this setting falls into the second case, gives $\overline{\chi}_b(\Pi_q) \leq (6-4\sqrt{2})v \approx 0.34v$.

3 Geometric lines in hypercubes

In this section, we consider the *n-cube over t elements*, denoted by C_t^n as an application of Theorem 1, where the set of *vertices* is the set of points in $\mathbb{R}^n$ with entries in $\{0,1,\ldots,t-1\}$ and the hyperedges are the geometric lines of C_t^n, that is, all the lines parallel to the axes and the main diagonals of maximal hyperplanes. More precisely, $C_t^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : 0 \leq x_i \leq t-1, x_i \in \mathbb{Z}\}$; and a set of t points in C_t^n is a geometric line if there is a labeling of the points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}$ such that if $\mathbf{x}_i = (x_{i,1}, x_{i,2}, x_{i,3}, \ldots, x_{i,n-2}, x_{i,n-1}, x_{i,n})$ for all $0 \leq i \leq t-1$, then for every $1 \leq j \leq n$ it holds that the entries of $(x_{0,j}, x_{1,j}, x_{2,j}, \ldots, x_{t-2,j}, x_{t-1,j})$ are all equal to some fixed value $a \in \{0,1,\ldots,t-1\}$; appear in increasing order $0,1,2,\ldots,t-1$; or appear in decreasing order $t-1, t-2, \ldots, 1, 0$.

It is clear, that the number of vertices of this hypergraph is $v = |(C_t^n)| = t^n$ and it is known that the number of hyperedges is $e = |\mathcal{L}(C_t^n)| = \frac{1}{2}((t+2)^n - t^n))$ [6]. Since any two points in the cube C_2^n are in a line, then $\overline{\chi}_b(C_2^n) = 1$. The general lower bound $\overline{\chi}_b(C_t^n) \geq \left(\frac{t}{2}\right)^n$ for any even $t \geq 4$ was proved in [9]. The following result is a direct application of Theorem 1 with $v = |C_t^n| = t^n$ and $e = |\mathcal{L}(C_t^n)| = ((t+2)^n - t^n)/2$.

Corollary 2. *Let t and n be positive integers. Then*

$$\overline{\chi}_b(C_t^n) \leq \begin{cases} \dfrac{3t^n-(t+2)^n}{2} & \text{if } t \geq \dfrac{2}{\sqrt[n]{2}-1}, \\ \left\lceil 2(t+2)^n - 2\sqrt{(t+2)^{2n}-t^{2n}} \right\rceil - 1 & \text{if } 2 \leq t < \dfrac{2}{\sqrt[n]{2}-1}. \end{cases}$$

It can be checked that when $t = 2$, this result implies $\overline{\chi}_b(C_2^n) = 1$. Recently, we have proved that this upper bound is tight when $t \geq 4n-2$ (see Theorem 3) by giving an intricate construction that uses the Hall's Marriage Theorem.

Theorem 3 ([2]). *For integers $n \geq 2$ and $t \geq 4n-2$, the balanced upper chromatic number of C_t^n is $\overline{\chi}_b(C_t^n) = \frac{3t^n-(t+2)^n}{2}$. This identity also holds for $(t, n) = (5, 2), (8, 3)$, and $(9, 3)$.*

We conjecture that Theorem 3 remains true for $2/(\sqrt[n]{2}-1) \leq t \leq 4n-2$. To confirm this, a rainbow-free coloring with classes of sizes 1 and 2 needs to be found. The inclusion of the cases $(t, n) = (5, 2), (8, 3)$, and $(9, 3)$ confirms this conjecture for dimensions 2 and 3. We now focus on the small values of t, namely, $2 < t < 2/(\sqrt[n]{2}-1)$. First, we present a *recursive* lower bound that will allow us to improve the lower bound in [9] and the best-known colorings in the space.

Theorem 4. *Let t and n be positive integers and suppose that t has a proper divisor $1 < d < t$. Then $\overline{\chi}_b(C_t^n) \geq \left(\frac{t}{d}\right)^n \overline{\chi}_b(C_d^n)$.*

Proof. (Sketch) Partition C_t^n into $(t/d)^n$ n-cubes over d elements. Color each of these smaller n-cubes with $\overline{\chi}_b(C_d^n)$ different colors for a total of $(t/d)^n\, \overline{\chi}_b(C_d^n)$. To prove that this coloring has no rainbow lines, we argue that any geometric line of C_t^n completely contains a geometric line of one of the $\left(\frac{t}{d}\right)^n$ copies of C_d^n. Since none of these smaller lines is rainbow (i.e., each of them has at least two points of the same color), then the larger line is not rainbow. □

When $t = 4$ and any n, the idea used in Theorem 4 can improve the lower bound in [9] for this case.

Proposition 5. *For $n \geq 2$, $\overline{\chi}_b(C_4^n) \geq 2^n + 1$.*

Proof. (Sketch) Partition the cube C_4^n into 2^n n-cubes over 2 elements $C_1, C_2, \ldots C_{2^n}$ and denote by C_0 the centered n-cube over 2 elements, that is $C_0 = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in C_4^n, \colon\ x_i \in \{1, 2\}\}$. Consider the sets $R_i = C_i - C_0$. Assign color 0 to all vertices in C_0 and color i to every vertex in R_i, $1 \leq i \leq 2^n$. Note that this is a balanced partition of C_4^n into $2^n + 1$ parts, 2^n of size $2^n - 1$ and one of size 2^n, (see Figure 1 for this coloring of C_4^n for $n = 3$). This coloring contains no rainbow lines. □

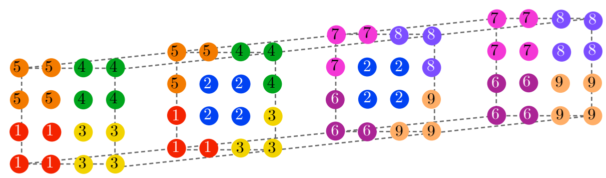

Figure 1: An illustration of the coloring in Proposition 5 for $n = 3$.

A direct application of Theorem 4 and Proposition 5 gives a lower bound that improves when t is a multiple of 4. Moreover, we were able to adapt the construction for every even t.

Theorem 6. *Let $n \geq 2$. If $2 \leq t \leq n$ and t is even, then $\overline{\chi}_b(C_t^n) \geq \left(\frac{t}{2}\right)^n + \left\lfloor \frac{t}{4} \right\rfloor^n$.*

Proof. (Sketch) If $t \equiv 0 \pmod 4$ the result follows directly by Theorem 4 and Proposition 5. If $t \equiv 2 \pmod 4$ then we use the construction for $t-2$ adapted as follows. For every $1 \leq i \leq n$ consider the two central $(n-1)$-dimentional hypercubes $C_{i,1}$ and $C_{i,2}$ defined as $C_{i,1} = \{(x_1, x_2, \ldots, x_n) \in C_t^n : x_i = \frac{t-2}{2}\}$ and $C_{i,2} = \{(x_1, x_2, \ldots, x_n) \in C_t^n : x_i = \frac{t}{2}\}$. Use the coloring provided by Theorem 4 and Proposition 5 for the set $C_t^n \setminus \cup_{i=1}^n (C_{i,1} \cup C_{i,2})$, which can be seen as a copy of C_{t-2}^n contained in C_t^n (see the top part of Figure 2(e) to visualize this copy of C_{t-2}^n contained in C_t^n when $(t,n) = (6,3)$, ignore the colors.) Then, only the lines contained in $\cup_{i=1}^n (C_{i,1} \cup C_{i,2})$ are not yet blocked by this partial coloring, but we can block them using copies of C_2^n, each of a different color. □

Note that this lower bound is nonsignificant for $t \geq 4n-2$ due to Theorem 3, but provides the best known bound for the remaining even values of t. The other modular classes of $t \bmod n$ would require a more detailed analysis that strongly depends on the dimension to expand the coloring of C_{t-1}^n to C_t^n. We illustrate this approach in Figure 2(d) for the case $(t,n) = (5,3)$.

4 A summary of exact results and best bounds

In this section, we summarize the best-known bounds for the cases $n = 2$ and $n = 3$. In the 2-dimentional case, the balanced upper chromatic number is completely determined. In the 3-dimentional case, four cases $4 \leq t \leq 7$ remain open. Table 1 shows the best bounds we know for these values.

Theorem 7. *For $n = 2$, $\overline{\chi}_b(C_3^2) = 2$, $\overline{\chi}_b(C_4^2) = 7$, and $\overline{\chi}_b(C_t^2) = t^2 - 2t - 2$, for $t \geq 5$. For $n = 3$, $\overline{\chi}_b(C_3^3) = 3$, and $\overline{\chi}_b(C_t^3) = t^3 - 3t^2 - 6t - 4$, for $t \geq 8$.*

Proof. In the plane $(n = 2)$, the cases $t \geq 5$ are covered by Theorem 3. The balanced rainbow-free 3-coloring shown in Figure 2(a) shows that the upper bound in Theorem 1 is tight for $t = 4$. It is known that in any balanced 3-coloring of C_3^2, there is a rainbow line, which means that $\overline{\chi}_b(C_3^2) \leq 2$ [9]. Since 2 colors are not enough to block a line in this cube, then $\overline{\chi}_b(C_3^2) = 2$. In the space $(n = 3)$, the cases $t \geq 8$ are covered by Theorem 3. For $t = 3$, the balanced rainbow-free 3-coloring shown in Figure 2(b) shows that $\overline{\chi}_b(C_3^3) \geq 3$. We ran a computer program to check that all balanced 4-colorings of C_3^3 contain a rainbow line showing that $\overline{\chi}_b(C_3^3) = 3$. Our program searched among a reduced set of $O(13!)$ colorings. Such a coloring would have 3 colors that are used 7 times and one color that is used 6 times. We reduced the number of possible colorings to be checked by fixing the color of the point in the center of the cube and using the fact that the other two points in any line through the center are either the same color or one of them is the same color as the center. □

The question of determining $\overline{\chi}_b(C_t^n)$ for $3 \leq t < 4n-2$ remains open for higher dimensions.

t	lower bound Th. 6	lower bound Fig. 2	upper bound Th. 2
4	4	12	18*
5	-	26	47
6	28	40	95
7	-	64	171

Table 1: Best bounds for $\overline{\chi}_b(C_t^3)$ when $n = 3$ and $4 \leq t \leq 7$. * The bound resulting from Theorem 2 is 19 but we have improved it to 18 [2].

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Figure 2: Rainbow-free colorings of (a) C_4^2, (b) C_3^3, (c) C_4^3. (d) C_5^3, (e) C_6^3, (f) C_7^3. The shaded regions highlight a coloring of a smaller size (e.g. the shaded region in (e) corresponds to the coloring in (c).) All colorings in (d)-(f) expand the one in (c), so (c)-(f) have color classes of sizes 5 and 6.

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