Sidorenko-type inequalities for Trees
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Abstract

Given two graphs H and G, the homomorphism density $t(H, G)$ represents the probability that a random mapping from $V(H)$ to $V(G)$ is a homomorphism. Sidorenko Conjecture states that for any bipartite graph H, $t(H, G)$ is greater or equal than $t(K_2, G)^{e(H)}$ for every graph G.

Introducing a binary relation $H \succcurlyeq T$ if and only if $t(H, G)^{e(T)} \geq t(T, G)^{e(H)}$ for all graphs G, we establish a partial order on the set of non-empty connected graphs. Employing a technique by Kopparty and Rossman [10], which involves the use of entropy to define a linear program, we derive several necessary and sufficient conditions for two trees T, F to satisfy $T \succcurlyeq F$. Furthermore, we show how important results and open problems in extremal graph theory can be reframed using this binary relation.

1 Introduction

One of the main objectives of extremal combinatorics is to study certain substructures in a large combinatorial object to understand the influence of local pattern frequencies on a global structure. This topic links many active areas of research, including the study of quasirandomness pioneered by Rödl [14], Thomason [17] and Chung, Graham and Wilson [4], the theory of combinatorial limits developed by Lovász and his collaborators, see [12], and the area of property testing in computer science spearheaded by Goldreich, Goldwasser and Ron [8].

A homomorphism from a graph H to a graph G is a function $f : V(H) \to V(G)$ such that $f(u)f(v) \in E(G)$ whenever $uv \in E(H)$. We denote by $Hom(H, G)$ the set of all possible homomorphisms between H and G. Let us denote $hom(H, G) = |Hom(H, G)|$. The homomorphism density, $t(H, G)$, is the probability that a random function $f : V(H) \to V(G)$ is a homomorphism.

$$t(H, G) = \frac{hom(H, G)}{v(G)^{v(H)}}.$$

Our focus is on proving inequalities for homomorphism densities of the following form:

$$t(F_2, G) \geq t(F_1, G)^{\alpha} \quad (1)$$

where F_1 and F_2 are fixed graphs, $\alpha > 0$ and the inequality in (1) holds for every graph G. Inequalities of this form are known as Sidorenko-type inequalities and several problems in extremal combinatorics

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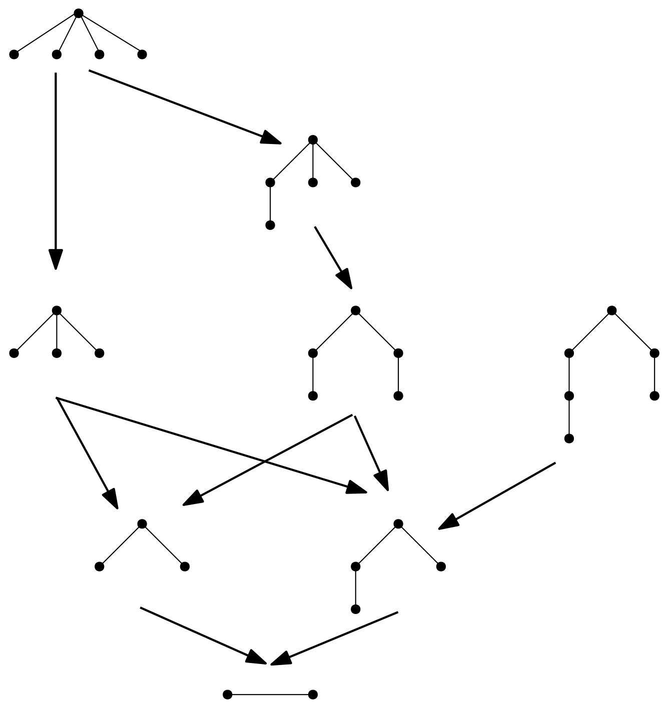

Figure 1: A section of the poset for trees.

can be expressed in terms of these inequalities. For example, the well-known Sidorenko Conjecture [16] states that $t(H,G) \geq t(K_2,G)^{|E(H)|}$ for every bipartite graph H. A systematic study of Sidorenko-type inequalities for graph homomorphisms via the method of tropicalization was recently initiated in [2, 3]. In [10], Kopparty and Rossman introduced a powerful method for using the information theoretic notion of entropy together with linear programming to prove Sidorenko-type inequalities. This approach is akin to the entropy-based approach which has seen recent success in the study of Sidorenko's Conjecture [5, 6] and other related problems [9, 11]. It was also used by Blekherman and Raymond [1] to give an illuminating alternative proof of the result of Sağlam [15] that

$$t(P_{k+2},G) \geq t(P_k,G)^{\frac{k+1}{k-1}} \tag{2}$$

for all $k \geq 2$ where, for all $\ell \geq 1$, P_ℓ denotes the path with ℓ vertices and $\ell - 1$ edges. This inequality was first conjectured by Erdős and Simonovits [7]. For a recent generalization of this result, see [2, Theorem 1.3].

Given two non-empty graphs H and T, we write $H \succcurlyeq T$ to mean that $t(H,G)^{e(T)} \geq t(T,G)^{e(H)}$ for every graph G. This binary relation is a partial order on the set of non-empty connected graphs. In Figure 1 we show the poset of some small trees.

2 The linear program.

Following the method introduced by Kopparty and Rossman, we reduce the problem of proving that $H \succcurlyeq T$ for forests H and T to solving a linear program. We obtained the full structure of the partial order on all pairs of trees with at most 8 vertices. Also, we characterize trees H such that $H \succcurlyeq S_k$ and $H \succcurlyeq P_4$, where S_k is the star on k vertices and P_4 is the path on 4 vertices.

Let $LP(H,T)$ be the following linear program. Let $\{w(\varphi) : \varphi \in Hom(H,T)\}$ be the variables.

$$\begin{aligned}
\text{maximize} \quad & \sum_{e\in E(T)} \sum_{\varphi\in Hom(H,T)} \mu_\varphi(e)\cdot w(\varphi) & \\
\text{subject to} \quad & \sum_{\varphi\in Hom(H,T)} \mu_\varphi(e)\cdot w(\varphi) \le 1 & \forall e \in E(T),\\
& \sum_{\varphi\in Hom(H,T)} \mu_\varphi(v)\cdot w(\varphi) \le 1 & \forall v \in V(T),\\
& w(\varphi) \ge 0 & \forall \varphi \in \mathrm{Hom}(H,T).
\end{aligned}$$

Where $\mu_\varphi(v) = |\varphi^{-1}(v)|$ for each $v \in V(T)$ and $\mu_\varphi(e) = |\varphi^{-1}(e)|$ for each $e \in E(T)$.

Lemma 1. *If H and T are forests such that the value of $LP(H,T)$ is equal to $e(T)$, then $H \succcurlyeq T$.*

We also define the dual of the linear program. Let $DLP(H,T)$ be the dual of $LP(H,T)$ with variables $\{y(m) : m \in V(T)\cup E(T)\}$ defined as follows:

$$\begin{aligned}
\text{minimize} \quad & \sum_{v\in V(T)} y(v) + \sum_{e\in E(T)} y(e) & \\
\text{subject to} \quad & \sum_{v\in V(T)} \mu_\varphi(v)\cdot y(v) + \sum_{e\in E(T)} \mu_\varphi(e)\cdot y(e) \ge e(H) & \forall \varphi \in \mathrm{Hom}(H,T),\\
& y(v) \ge 0 \qquad \forall v \in V(T) & \\
& y(e) \ge 0 \qquad \forall e \in E(T). &
\end{aligned}$$

Lemma 2. *If H and T are non-empty graphs such that the value of $DLP(H,T)$ is less than $e(T)$, then $H \not\succcurlyeq T$.*

3 Main results.

Given two trees H and T, the following theorems give sufficient or necessary conditions for $H \succcurlyeq T$. We let $\sigma(H)$ be the minimum of $|A|, |B|$ in the bipartition (A,B) for the tree.

Theorem 3. *If $H \succcurlyeq T$, then*

$$\frac{e(H)}{\sigma(H)} \ge \frac{e(T)}{\sigma(T)}.$$

The last theorem holds for any H and T bipartite graphs. For the sufficient condition, we say that a *fractional orientation* of a graph T is a function $f : V(T)\times V(T) \to [0,\infty)$ such that $f(u,v)+f(v,u) = 1$ for any edge $uv \in E(T)$ and $f(u,v) = 0$ if $uv \notin E(T)$.

The *out-degree* and *in-degree* of a vertex $v \in V(T)$ are $d_f^+(v) := \sum_{u\in V(T)} f(v,u)$ and $d_f^-(v) := \sum_{u\in V(T)} f(u,v)$, respectively.

Theorem 4. *If there exists a fractional orientation of T such that, for all $v \in V(T)$,*

$$\frac{d_f^-(v)\cdot(v(H)-\sigma(H)) + d_f^+(v)\cdot\sigma(H)}{e(H)} \le 1, \tag{3}$$

then $H \succcurlyeq T$.

Using Theorem 3 and 4, we get the characterization for stars.

Corollary 5. *Let $k \ge 3$ and let H be a non-empty tree. Then $H \succcurlyeq S_k$ if and only if $e(H) \ge (k-1)\sigma(H)$.*

Finally, the following gives a characterization for P_4.

Theorem 6. *Let H be a tree. Then $H \succcurlyeq P_4$ if and only if H has at least four vertices.*

Nevertheless, it is not easy to generalize the result of Theorem 6 to a more general case.

Theorem 7. *Let H be a k-vertex near-star with ℓ leaves. If $\frac{k+1}{2} \leq \ell \leq k-3$, then $H \not\succcurlyeq P_k$.*

We believe that the following weaker generalization may hold. This statement, if true, would support the rough intuition that path-like graphs are near the bottom of the partial order restricted to trees.

Conjecture 8. *For any $k \geq 1$, there exists $n_0(k)$ such that if H is a tree with at least $n_0(k)$ vertices, then $H \succcurlyeq P_{2k}$.*

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