On the sum of several finite subsets in $\mathbb{R}^2$ *

Mario Huicochea[†1], René González-Martínez [‡2], Amanda Montejano[§3], and David Suárez[¶3]

[1]Universidad Autónoma de San Luis Potosí
[2]Universidad Autónoma de Zacatecas
[3]UMDI, Facultad de Ciencias, UNAM Juriquilla, Querétaro, México

Abstract

Let $h \geq 2$ be an integer, $\mathbf{u} \in \mathbb{R}^2 \setminus \{(0,0)\}$, and $A_1, A_2, \ldots, A_h$ be nonempty finite subsets of $\mathbb{R}^2$. For each $i \in \{1, 2, \ldots, h\}$, denote by m_i the number of lines parallel to the line generated by the vector $\mathbf{u}$ that intersect A_i. We show that

$$|A_1 + A_2 + \ldots + A_h| \geq \left(\left(\sum_{i=1}^{h} \frac{|A_i|}{m_i} \right) - (h-1) \right) \left(\left(\sum_{i=1}^{h} m_i \right) - (h-1) \right)$$

generalizing a statement of D. J. Grynkiewicz and O. Serra for $h = 2$. We also characterize the case of equality; that is, we describe the structure of finite 2-dimensional subsets of $\mathbb{R}^2$ which are extremal with respect to the inequality above. This also generalizes a result of G. A Freiman, D. Grynkiewicz, O. Serra and Y. V. Stanchescu.

1 Introduction

One of the most important problems in Additive Number Theory has been to determine nontrivial lower bounds for the cardinality of $A_1 + A_2 + \ldots + A_h = \{\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_h \mid \mathbf{a}_i \in A_i \text{ for each } i \in \{1, 2, \ldots, h\}\}$, where $A_1, A_2, \ldots, A_h$ are nonempty finite subsets of an abelian group G, see for instance [2, 7, 9, 11, 12, 13]. Particularly, there is interesting recent progress concerning this problem in $\mathbb{R}^d$. In this work, we only focus our attention in $\mathbb{R}^2$. Given $\mathbf{u} \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $A_1, A_2, \ldots, A_h$ nonempty finite subsets of $\mathbb{R}^2$, we give a lower bound of $|A_1 + A_2 + \ldots + A_h|$ in terms of the number of lines parallel to the line generated by $\mathbf{u}$ which intersect A_i, for each $i \in \{1, 2, \ldots, h\}$. This was already done for two sets ($h = 2$) by Grynkiewicz and Serra [10]. Moreover, Freiman, Grynkiewicz, Serra and Stanchescu characterized the extremal 2-dimensional sets attaining such lower bound [5]. We generalize both results for any integer $h \geq 2$.

Given $\mathbf{u} \in \mathbb{R}^2$, we denote by $\langle \mathbf{u} \rangle$ the subspace (line) generated by $\mathbf{u}$. Let $\phi_{\langle \mathbf{u} \rangle} : \mathbb{R}^2 \to \mathbb{R}^2 / \langle \mathbf{u} \rangle$ the natural projection modulo $\langle \mathbf{u} \rangle$. For a finite subset A of $\mathbb{R}^2$, let $\phi_{\langle \mathbf{u} \rangle}(A) = \{\phi_{\langle \mathbf{u} \rangle}(\mathbf{a}) \mid \mathbf{a} \in A\}$. Thus, if $\mathbf{u} \neq (0,0)$, $|\phi_{\langle \mathbf{u} \rangle}(A)|$ is the number of lines parallel to $\langle \mathbf{u} \rangle$ that intersect A. As we already mentioned in the previous paragraph, Grynkiewicz and Serra were able to find a lower bound of $|A+B|$ for nonempty subsets A and B of $\mathbb{R}^2$ in terms of $|\phi_{\langle \mathbf{u} \rangle}(A)|$ and $|\phi_{\langle \mathbf{u} \rangle}(B)|$. Here we give the precise statement.

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†Email: dym@cimat.mx
‡Email: reneglzmtz@gmail.com
§Email: amandamontejano@ciencias.unam.mx.
¶Email: suardavid@hotmail.com

Theorem 1 (Grynkiewicz-Serra). *Let A and B be nonempty finite subsets of $\mathbb{R}^2$. For every $\mathbf{u} \in \mathbb{R}^2 \setminus \{(0,0)\}$,*

$$|A+B| \geq \left(\frac{|A|}{m} + \frac{|B|}{n} - 1\right)(m+n-1), \tag{1}$$

where $m = |\phi_{\langle \mathbf{u} \rangle}(A)|$ and $n = |\phi_{\langle \mathbf{u} \rangle}(B)|$.

Proof. see [10, Thm. 1.3]. □

We generalize Theorem 1 for an arbitrary number of nonempty finite subsets of $\mathbb{R}^2$.

Theorem 2. *Let $h \geq 2$ be an integer, and let $A_1, \dots A_h$ be nonempty finite subsets of $\mathbb{R}^2$. For every $\mathbf{u} \in \mathbb{R}^2 \setminus \{(0,0)\}$,*

$$|A_1 + A_2 + \ldots + A_h| \geq \left(\left(\sum_{i=1}^{h} \frac{|A_i|}{m_i}\right) - (h-1)\right)\left(\left(\sum_{i=1}^{h} m_i\right) - (h-1)\right) \tag{2}$$

where $m_i = |\phi_{\langle \mathbf{u} \rangle}(A_i)|$ for each $i \in \{1, 2, \dots, h\}$.

In order to characterized the extremal sets attaining equality in (2) we need to define some sets called *trapezoids*. For the sake of clarity, we will use a definition that varies slightly from the one originally presented in [5].

Definition 3. *Let $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle$ be an ordered base of $\mathbb{R}^2$, and let $m, h, c \in \mathbb{Z}$ with $(h-1) + (m-1)c \geq 0$. We say that a finite nonempty 2-dimensional set $A \subset \mathbb{R}^2$ is a* trapezoid of type $T_{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle}(m, h, c)$ *if there is a vector $\boldsymbol{v} \in \mathbb{R}^2$ such that*

$$M_{\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle}(A) + \boldsymbol{v} = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq m-1,\ 0 \leq x \leq (h-1) + cy\},$$

where $M_{\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle} : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear mapping that leads the ordered base $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle$ to the canonical ordered base $\langle \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle$.

We shall note that the example showed in [5] as a standard trapezoid $T(6, 19, 2, -1)$ correspond to the translation of any trapezoid of type $T_{\langle (1,2),(0,1) \rangle}(19, 6, -3)$, see Figure 1. In general, a standard trapezoid $T(m, h, c, d)$ corresponds, after applying the linear transformation given by the matrix $M_{\langle (1,d),(0,1) \rangle} = \begin{pmatrix} -d & 1 \\ 1 & 0 \end{pmatrix}$, to a translation of any trapezoid of type $T_{\langle (1,d),(0,1) \rangle}(m, h, d-c)$.

Theorem 4. *Let $A_1 \dots, A_k$ be nonempty finite 2-dimensional subsets of $\mathbb{R}^2$, and let $\boldsymbol{u} \in \mathbb{R}^2$. If*

$$|A_1 + \cdots + A_k| = \left(\sum_{i=1}^{k} \left(\frac{|A_i|}{m_i}\right) - (k-1)\right)\left(\sum_{i=1}^{k} (m_i) - (k-1)\right), \tag{3}$$

where $m_i = |\phi_{\langle \boldsymbol{u} \rangle}(A_i)|$ for $1 \leq i \leq k$, then each A_i is a trapezoid of type $T_{\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle}(m_i, h_i, c)$ for some ordered base $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle$, and some integers $h_1, \dots, h_k$, with common slope c.

The paper is organized as follows: Section 2 contains auxiliary results needed for proving Theorems 2 and 4. In particular, we present some properties of the technique known as (linear) compression. The proof of Theorem 2 is completed in Section 3. To prove Theorem 4 is a bit more technical. We present the strategy in Section 4.

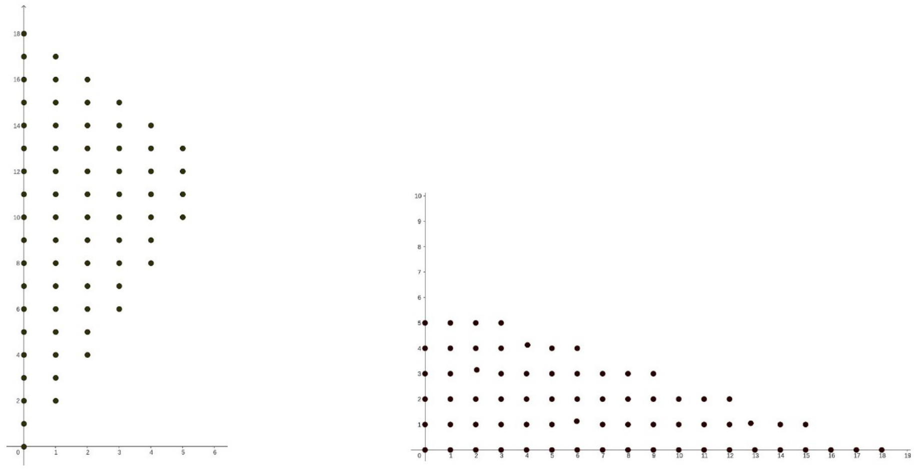

Figure 1: The standard trapezoid $T(6,19,2,-1)$ (depicted on the left) given as an example in [5] corresponds to the trapezoid of type $T_{\langle(1,2),(0,1)\rangle}(6,19,-3)$ translated to the origin (depicted on the right). This can be seen by applying the linear transformation $M_{\langle(1,2),(0,1)\rangle}$ to $T(6,19,2,-1)$.

2 Preliminaries

Let $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ be an ordered basis of $\mathbb{R}^2$. For a finite subset $A \subset \mathbb{R}^2$ and $i \in \{1,2\}$, the *linear compression* of A with respect to $\mathbf{u}_i$, denoted by $\mathbf{C}_i(A)$, is defined as follows. Take $j \in \{1,2\} \setminus \{i\}$ and let $\mathbf{C}_i(A)$ be the set satisfying that for each $\mathbf{x} \in \langle \mathbf{u}_j \rangle$,

$$\phi_{\langle \mathbf{u}_j \rangle}\left(\mathbf{C}_i(A) \cap (\langle \mathbf{u}_i \rangle + \mathbf{x})\right) = \{0, \mathbf{u}_i, 2\mathbf{u}_i, \cdots, (r-1)\mathbf{u}_i\} + \langle \mathbf{u}_j \rangle,$$

where $r = |A \cap (\langle \mathbf{u}_i \rangle + \mathbf{x})\,|$ and, if $r = 0$, we consider $\mathbf{C}_i(A) \cap (\langle \mathbf{u}_i \rangle + \mathbf{x}) = \emptyset$. The *compression of A* with respect to the ordered basis $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ is then defined by $\mathbf{C}(A) = \mathbf{C}_2(\mathbf{C}_1(A))$. Several properties of compression can be found in [1, 6, 8, 10]; we just need a few of them. Observe that we have by definition that

$$|A| = |\mathbf{C}(A)|, \tag{4}$$

and

$$|\phi_{\langle \mathbf{u}_1 \rangle}(A)| = |\phi_{\langle \mathbf{u}_1 \rangle}(\mathbf{C}(A))|. \tag{5}$$

One of the main properties of compression is the following.

Lemma 5. *For any nonempty finite subsets $A_1, A_2 \subset \mathbb{R}^2$, and an ordered basis $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle$ of $\mathbb{R}^2$, it happens that $\mathbf{C}(A_1 + A_2) \supseteq \mathbf{C}(A_1) + \mathbf{C}(A_2)$. In particular, $|A_1 + A_2| \geq |\mathbf{C}(A_1) + \mathbf{C}(A_2)|$.*

Proof. See [6, Lemma 3.4]. □

We will also make use of the following well known fact.

Theorem 6 (Folklore)**.** *Let $A_1, \ldots, A_h$ be finite nonempty subsets of a torsion-free abelian group. Then*

$$|A_1 + A_2 + \ldots + A_h| \geq \left(\sum_{i=1}^{h} |A_i| \right) - (h-1),$$

and the equality is achieved when $A_1, A_2, \ldots, A_h$ are arithmetic progressions with the same common difference.

Proof. See for instance [11, Theorem 1.4]. □

Let $A_1, A_2, \ldots, A_h$ be nonempty finite subsets of $\mathbb{R}^2$ and, for each $i \in \{1, 2, \ldots, h\}$, let $\mathbf{C}(A_i)$ be the compression of A_i with respect to the ordered basis $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$. The projection $\phi_{\langle \mathbf{u}_1 \rangle}$ is a linear mapping, and the definition of $\mathbf{C}(A_i)$ implies that $\phi_{\langle \mathbf{u}_1 \rangle}(\mathbf{C}(A_i))$ is an arithmetic progression with difference $\mathbf{u}_2$ for each $i \in \{1, 2, \ldots, h\}$. From these facts and Theorem 6, it follows that

$$\left|\phi_{\langle \mathbf{u}_1 \rangle}\left(\mathbf{C}(A_1) + \mathbf{C}(A_2) + \ldots + \mathbf{C}(A_h)\right)\right| = \left(\sum_{i=1}^{h} \left|\phi_{\langle \mathbf{u}_1 \rangle}\left(\mathbf{C}(A_i)\right)\right|\right) - (h-1). \tag{6}$$

In order to prove Theorem 2 , we prove the next inequality.

Lemma 7. *Let $A_1, A_2, \ldots, A_h$ be nonempty finite subsets of $\mathbb{R}^2$, and let $\mathbf{C}(A_i)$ be the compression of A_i with respect to the ordered basis $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle$. Then,*

$$|A_1 + A_2 + \ldots + A_h| \geq |\mathbf{C}(A_1) + \mathbf{C}(A_2) + \ldots + \mathbf{C}(A_h)|. \tag{7}$$

Proof. By induction on h taking $A = A_1 + A_2 + \ldots + A_{h-1}$ and A_h, with the use of Lemma 5. □

3 Proof of Theorem 2

Proof. We proceed by induction on h. If $h = 2$, the statement follows by Theorem 1. Consider now the sets $A = \mathbf{C}(A_1) + \mathbf{C}(A_2) + \ldots + \mathbf{C}(A_{h-1})$ and $B = \mathbf{C}(A_h)$ where, for each $i \in \{1, 2, \ldots, h\}$, $\mathbf{C}(A_i)$ is the compression of A_i with respect to the ordered basis $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}, \mathbf{u}^{\perp} \rangle$. Set $m = |\phi_{\langle \mathbf{u} \rangle}(A)|$ and $n = |\phi_{\langle \mathbf{u} \rangle}(B)|$. Then

$$\begin{aligned}
|A_1 + \ldots + A_h| &\geq |\mathbf{C}(A_1) + \ldots + \mathbf{C}(A_{h-1}) + \mathbf{C}(A_h)| && \Big(\text{by Lemma 7}\Big) \\
&= |A + B| \\
&\geq \left(\frac{|A|}{m} + \frac{|B|}{n} - 1\right)(m + n - 1). && \Big(\text{by Thm. 1}\Big)
\end{aligned} \tag{8}$$

Hence, by definition and (5),

$$n = |\phi_{\langle \mathbf{u} \rangle}(B)| = |\phi_{\langle \mathbf{u} \rangle}(\mathbf{C}(A_h))| = |\phi_{\langle \mathbf{u} \rangle}(A_h)| = m_h, \tag{9}$$

and

$$\begin{aligned}
m = |\phi_{\langle \mathbf{u} \rangle}(A)| &= |\phi_{\langle \mathbf{u} \rangle}(\mathbf{C}(A_1) + \mathbf{C}(A_2) + \ldots + \mathbf{C}(A_{h-1}))| \\
&= \left(\sum_{i=1}^{h-1} \left|\phi_{\langle \mathbf{u} \rangle}\left(\mathbf{C}(A_i)\right)\right|\right) - (h-2) && \Big(\text{by (6)}\Big) \\
&= \left(\sum_{i=1}^{h-1} \left|\phi_{\langle \mathbf{u} \rangle}\left(A_i\right)\right|\right) - (h-2) && \Big(\text{by (5)}\Big) \\
&= \left(\sum_{i=1}^{h-1} m_i\right) - (h-2).
\end{aligned} \tag{10}$$

Thus (9) and (10) yield that

$$m + n - 1 = \left(\sum_{i=1}^{h-1} m_i\right) - (h-2) + m_h - 1 = \left(\sum_{i=1}^{h} m_i\right) - (h-1). \tag{11}$$

By (4) we know $|B| = |\mathbf{C}(A_h)| = |A_h|$, and so, by (9), we get

$$\frac{|B|}{n} = \frac{|A_h|}{m_h}. \tag{12}$$

Now, since (4) and (5) imply $|\mathbf{C}(A_i)| = |A_i|$ and $|\phi_{\langle \mathbf{u} \rangle}(\mathbf{C}(A_i))| = |\phi_{\langle \mathbf{u} \rangle}(A_i)| = m_i$, it follows by definition and the induction hypothesis that

$$|A| = |\mathbf{C}(A_1) + \mathbf{C}(A_2) + \ldots + \mathbf{C}(A_{h-1})| \geq \left(\left(\sum_{i=1}^{h-1} \frac{|A_i|}{m_i}\right) - (h-2)\right)\left(\left(\sum_{i=1}^{h-1} m_i\right) - (h-2)\right). \quad (13)$$

Thus, (10) and (13) imply

$$\frac{|A|}{m} \geq \left(\sum_{i=1}^{h-1} \frac{|A_i|}{m_i}\right) - (h-2). \quad (14)$$

Finally, substituying (11), (12) and (14) in (8), we obtain

$$\begin{aligned} |A_1 + \ldots + A_h| &\geq \left(\frac{|A|}{m} + \frac{|B|}{n} - 1\right)(m+n-1) \\ &\geq \left(\left(\sum_{i=1}^{h-1} \frac{|A_i|}{m_i}\right) - (h-2) + \frac{|A_h|}{m_h} - 1\right)\left(\left(\sum_{i=1}^{h} m_i\right) - (h-1)\right), \\ &= \left(\left(\sum_{i=1}^{h} \frac{|A_i|}{m_i}\right) - (h-1)\right)\left(\left(\sum_{i=1}^{h} m_i\right) - (h-1)\right), \end{aligned}$$

and the prove is completed. □

4 Sketch of the proof of Theorem 4

Observe that, if A is trapezoid of type $T_{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle}(m, h, c)$ for some ordered base $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ of $\mathbb{R}^2$, and some integers m, h and c satisfying $(h-1) + (m-1)c \geq 0$ then, by definition, there is a vector $\mathbf{v}$ such that

$$M_{\langle \mathbf{u}_1, \mathbf{u}_2 \rangle} A + \mathbf{v} = \bigcup_{i=0}^{m-1} \{(x, i) | 0 \leq x \leq (h-1) + ci\}.$$

Therefore,

$$|A| = m\left(h + \frac{c(m-1)}{2}\right). \quad (15)$$

With the use of (15) it is not hard to prove, by induction on k, the following.

Lemma 8. *Let $A_1, \ldots, A_k$ be trapezoids of type $T_{\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle}(m_i, h_i, c)$ for some ordered base $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle$, integers m_i and h_i, for each $1 \leq i \leq h$, and a common slope c. Then $A_1 + \cdots + A_k$ is a trapezoid of type $T_{\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle}(\sum_{i=1}^{k} m_i - (k-1), \sum_{i=1}^{k} h_i - (k-1), c)$.*

One of the key parts of the proof of Theorem 4 was to generalize a beautiful lemma which was used to prove Theorem 1 as well as the characterization of the extremal cases, see [10, 5]. For the sake of clarity, we present here only the statement for $k = 3$ and a sketch of its proof.

Lemma 9. *Let I, J, K be nonempty finite subsets of $\mathbb{R}$ with $min(|I|, |J|, |K|) \geq 2$. Let $a = \{a_i\}_I$, $b = \{b_j\}_J$ and $c = \{c_k\}_K$ sequences with $a_i, b_j, c_k > 0$ for $i \in I$, $j \in J$ and $k \in K$. For each $t \in I + J + K$, let $u_t(a, b, c) = max\{a_i + b_j + c_{k-i-j} : i \in I, j \in J, k - i - j \in K\}$. If*

$$\frac{1}{|I|}\sum_{i \in I} a_i + \frac{1}{|J|}\sum_{j \in J} b_j + \frac{1}{|K|}\sum_{k \in K} c_k \leq \frac{1}{|I| + |J| + |K| - 2} \sum_{t \in I+J+K} u_t(a, b, c). \quad (16)$$

If the equality holds then I, J and K are arithmetic progressions with common difference and the sequences a, b and c are also arithmetic progressions with common difference.

Proof. (sketch) For a sequence $\{x_i\}_{i\in L}$, denote by $\overline{x} = \frac{1}{|L|}\sum_{i\in L} x_i$ its average value. If $y = \{y_i\}_{i\in M}$ and $z = \{y_i\}_{i\in N}$ are also sequences, denote by $u^+(x,y,z)$ the subsequence of the $|L|+|M|+|N|-2$ elements in the sequence $u(x,y,z) = \{u_t(x,y,z) : t \in L+M+N\}$ which is well-defined in view of Theorem 6. Let $d = \{u_t(b,c) : t \in J+K\}$. First we shall prove that $u(a,b,c) = u(a,d)$ and then we need to prove that $\overline{u^+(a,d^+)} \le \overline{u^+(a,d)}$, which will lead us to show that

$$\overline{u^+(a,b,c)} \le \left(\frac{1}{|I|+|J|+|K|-2}\right) \sum_{t\in I+J+K} u_t(a,b,c). \tag{17}$$

From this position it is not hard to prove that

$$\frac{1}{|I|}\sum_{i\in I} a_i + \frac{1}{|J|}\sum_{j\in J} b_j + \frac{1}{|K|}\sum_{k\in K} c_k \le \frac{1}{|I|+|J|+|K|-2}\sum_{t\in I+J+K} u_t(a,b,c) \tag{18}$$

Now, suppose that the equality holds if (16), we can see that I, J and K are arithmetic progressions with common difference. From here, one has to work to show that actually, the sequences a, b and c are arithmetic progression with a common difference. □

To prove Theorem 4 we define one set for each $1 \le i \le k$ as $I_i = \phi_{\langle \mathbf{u}\rangle}(A_i)$, and work to obtain the base $\langle \mathbf{u}_1, \mathbf{u}_2\rangle$ and the parameters of the trapezoids in terms of the differences of the arithmetic progression given by Lemma 9.

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