# Disconnected Common Graphs via Supersaturation*

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**Abstract**

A graph $H$ is said to be *common* if the number of monochromatic labelled copies of $H$ in a 2-colouring of the edges of a large complete graph is asymptotically minimized by a random colouring. It is well known that the disjoint union of two common graphs may be uncommon; e.g., $K_2$ and $K_3$ are common, but their disjoint union is not. We investigate the commonality of disjoint unions of multiple copies of $K_3$ and $K_2$. As a consequence of our results, we obtain an example of a pair of uncommon graphs whose disjoint union is common. Our approach is to reduce the problem of showing that certain disconnected graphs are common to a constrained optimization problem in which the constraints are derived from supersaturation bounds related to Razborov's Triangle Density Theorem. We also improve bounds on the Ramsey multiplicity constant of a triangle with a pendant edge and the disjoint union of $K_3$ and $K_2$.

## 1 Introduction

In one of the first applications of the probabilistic method, Erdős [6] showed that a random colouring of the edges of a clique on $(1-o(1))2^{-1/2}e^{-1}k2^{k/2}$ vertices with red and blue contains no monochromatic complete graph on $k$ vertices with positive probability; this implies a lower bound on the *Ramsey number* of the complete graph $K_k$, i.e. the smallest $N$ for which every 2-colouring of the edges of $K_N$ contains a monochromatic $K_k$. To this day, Erdős' bound has been improved only slightly by Spencer [24]. One of the core themes in Ramsey theory is that random colourings tend to perform well in avoiding certain monochromatic substructures.

This intuition extends to the closely related area of "Ramsey multiplicity" in which the goal is to minimize the number of monochromatic labelled copies of a given graph $H$ in a red/blue colouring of the edges of $K_N$ asymptotically as $N$ tends to infinity. A graph $H$ is said to be *common* if this asymptotic minimum is achieved by a sequence of random colourings. A famous result of Goodman [10] implies that $K_3$ is common (see Theorem 7). Inspired by this, Erdős [5] conjectured that $K_k$ is common for all $k$ and, nearly two decades later, Burr and Rosta [4] conjectured that every graph $H$ is common. Sidorenko [22] observed that the *paw graph* $P$ consisting of a triangle with a pendant edge is uncommon. Around the same time, Thomason [25] showed that $K_k$ is uncommon for all $k \geq 4$; thus, the aforementioned conjectures are both false. Later, Jagger, Sťovíček and Thomason [14] proved that every graph $H$ containing a $K_4$ is uncommon. In particular, almost every graph is uncommon. In recent years, there has been a steady flow of results proving that the members of certain families of graphs are common or uncommon [16, 17, 11, 1, 2, 12, 15]. In spite of this, the task of classifying common graphs seems hopelessly difficult.

The main goal of this paper is to provide a new approach for bounding the number of monochromatic copies of certain disconnected graphs in a colouring of $K_N$ and to use it to obtain several new families

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of common graphs. Given graphs $H_1$ and $H_2$, let $H_1 \sqcup H_2$ denote their disjoint union; also, for a graph $F$ and $\ell \geq 1$, let $\ell \cdot F$ be the disjoint union of $\ell$ copies of $F$. The argument of Sidorenko [22] that the paw graph is uncommon also shows that $K_3 \sqcup K_2$ is uncommon (with the same proof). Most of our results focus on the commonality of unions of several copies of $K_3$ and $K_2$. Our first result is as follows.

**Theorem 1.** *For $0 \leq \ell \leq 2$, the graph $(2 \cdot K_3) \sqcup (\ell \cdot K_2)$ is common.*

We also show that this is best possible in the sense that $(2 \cdot K_3) \sqcup (3 \cdot K_2)$ is uncommon; see Proposition 10. Since $K_3$ and $K_2$ are both common, Sidorenko's result [22] that $K_3 \sqcup K_2$ is uncommon tells us that the disjoint union of two common graphs can be uncommon. Using Theorem 1, we find that the opposite phenomenon is also possible; the disjoint union of two uncommon graphs can be common. In fact, the disjoint union of two copies of a single uncommon graph can be common.

**Corollary 2.** *There exists an uncommon graph $H$ such that $H \sqcup H$ is common.*

*Proof.* Consider $H = K_3 \sqcup K_2$. The fact that $H$ is uncommon was shown by Sidorenko [22], and the fact that $H \sqcup H$ is common follows from Theorem 1 with $\ell = 2$. □

We remark that our results also allow us to obtain new examples of graphs $H_1$ and $H_2$ such that $H_1$ is common, $H_2$ is uncommon and $H_1 \sqcup H_2$ is common. However, the existence of such a pair of graphs was already known; see [17, Subsection 1.1]. We also prove a general result on disjoint unions of triangles and edges, provided that the number of triangles is at least three.

**Theorem 3.** *For $k \geq 3$ and $0 \leq \ell \leq 5k/3 (\approx 1.666k)$, the graph $(k \cdot K_3) \sqcup (\ell \cdot K_2)$ is common.*

**Theorem 4.** *For $k \geq 1$ and $\ell = \lceil 1.9665k \rceil$, the graph $(k \cdot K_3) \sqcup (\ell \cdot K_2)$ is uncommon.*

## 2 Preliminary

Several of the results in this paper are best understood in the context of graph limits. A *kernel* is a bounded measurable function $U : [0,1]^2 \to \mathbb{R}$ such that $U(x,y) = U(y,x)$ for all $x, y \in [0,1]$. A *graphon* is a kernel such that $0 \leq W(x,y) \leq 1$ for all $x, y \in [0,1]$. The set of all graphons is denoted $\mathcal{W}_0$. Given a graph $G$, let $v(G) := |V(G)|$ and $e(G) := |E(G)|$. A graph $G$ is said to be *empty* if $e(G) = 0$. Each graph $G$ can be associated to a graphon $W_G$ by dividing $[0,1]$ into $v(G)$ intervals $I_1, \dots, I_{v(G)}$ of equal measure corresponding to the vertices of $G$ and setting $W_G$ equal to 1 on $I_i \times I_j$ if the $i$th and $j$th vertices are adjacent and 0 otherwise. The *homomorphism density* of a graph $H$ in a kernel $U$ is defined by

$$t(H,U) := \int_{[0,1]^{V(H)}} \prod_{uv \in E(H)} W(x_u, x_v) dx_{V(H)}$$

where $x_{V(H)} = (x_v : v \in V(H))$. We refer the reader to [19] for more background on graph limits. The *Ramsey multiplicity constant* of a graph $H$ is defined to be

$$c(H) := \inf_{W \in \mathcal{W}_0} (t(H,W) + t(H, 1-W)).$$

In this language, a graph $H$ is *common* if and only if

$$c(H) = 2(1/2)^{e(H)}. \tag{1}$$

As stated above, $K_3 \sqcup K_2$ and the paw graph $P$ are uncommon. We obtain, to our knowledge, the tightest known upper bounds on the Ramsey multiplicity constants of these two graphs; for the former graph, we also obtain a reasonably tight lower bound which is proven without the assistance of the flag algebra method.

**Theorem 5.** $0.121423 < c(K_3 \sqcup K_2) < 0.121450.$

**Theorem 6.** *The paw graph $P$ satisfies $c(P) < 0.121415$.*

Note that, for every graph $H$ such that $c(H)$ is currently known, either $H$ is common or $c(H)$ is achieved by a "Turán graphon" $W_{K_k}$ for some $k \geq 3$ [8, 13]. To our knowledge, Theorem 5 is the closest that any result has come to determining $c(H)$ for a graph $H$ which does not fit into either of these two categories. The lower bound in Theorem 5 can be improved by at least 0.022% using the flag algebra method; however, such a proof would most likely be verifiable only with heavy computer assistance, and is thus unlikely to provide much in terms of valuable insights. Several of the known results on common graphs actually establish stronger inequalities than (1). Following [2], a non-empty graph $H$ is said to be *strongly common* if

$$t(H, W) + t(H, 1 - W) \geq t(K_2, W)^{e(H)} + t(K_2, 1 - W)^{e(H)} \tag{2}$$

for every graphon $W$. A simple application of Jensen's Inequality tells us that every strongly common graph is common. A classical example of a strongly common graph is $K_3$; see Theorem 7. A non-empty graph $H$ is said to be *Sidorenko* if

$$t(H, W) \geq t(K_2, W)^{e(H)} \tag{3}$$

for every graphon $W$. Clearly, every Sidorenko graph is strongly common which, in turn, implies that every such graph is common. By taking $W = W_{K_2}$, one can see that every Sidorenko graph must be bipartite. Sidorenko's Conjecture [23] famously states that every bipartite graph is Sidorenko. Currently, every bipartite graph $H$ which is known to be common is also known to be Sidorenko. Also, the only known examples of strongly common graphs which are not Sidorenko are the odd cycles [2, 10, 15].

Our strategy for obtaining new examples of common graphs relies on strong correlation inequalities, such as (2) and (3). Given this, it is natural to wonder whether all common graphs are strongly common; this question was raised in [2]. As it turns out, this is far from true. For example, $K_3 \sqcup K_3$ is common but not strongly common, and there are many other examples as well.

## 3 Key Ideas

Our approach is to reduce the problem of showing that certain disconnected graphs are common to a constrained optimization problem, in which the constraints are derived from supersaturation bounds related to Razborov's Triangle Density Theorem. For the purposes of proving the lower bound of Theorem 5, it will be enough to use the following theorem which was first announced by Fisher [7]; as mentioned in [21], the proof contained a hole that can be patched using a later result of [9]. A new proof was found by Razborov [20] prior to proving the general Triangle Density Theorem in [21].

**Theorem 7** (Goodman's Theorem [10])**.** *$K_3$ is strongly common.*

**Theorem 8** (Fisher [7] and Goldwurm and Santini [9]; see also Razborov [20])**.** *Every graphon $W$ with $t(K_2, W) \leq 2/3$ satisfies*

$$t(K_3, W) \geq \frac{1}{9}\left(-2\left(2 + \sqrt{4 - 6t(K_2, W)}\right) + 3t(K_2, W)\left(3 + \sqrt{4 - 6t(K_2, W)}\right)\right)$$

**Theorem 9** (Bollobás [3])**.** *Every graphon $W$ satisfies*

$$t(K_3, W) \geq \frac{4}{3}t(K_2, W) - \frac{2}{3}.$$

To prove Thoerem 4, the upper bound of Theorem 5 and Theorem 6, the graphons that we will use are all of the same general form. For $n \geq 1$, let $\triangle_n$ be the set of all vectors $\vec{z}$ of length $n$ with non-negative entries that sum to one. Given $\vec{z} \in \triangle_n$ and an $n \times n$ symmetric matrix $A$ with entries in

$[0,1]$, let $W_{\vec{z},A}$ be defined as follows. First, divide $[0,1]$ into $n$ intervals $I_1,\dots,I_n$ such that the measure of $I_i$ is equal to $\vec{z}_i$. Next, for each $1 \le i,j \le n$, define $W_{\vec{z},A}$ to be equal to $A_{i,j}$ for all $(x,y) \in I_i \times I_j$. It is easily observed that, for any graph $H$,

$$t(H, W_{\vec{z},A}) = \sum_{f:V(H)\to[n]} \prod_{v\in V(H)} \vec{z}_{f(v)} \prod_{uv\in E(H)} A_{f(u),f(v)}. \tag{4}$$

Using this construction, we could prove Theorem 4. Let $k \ge 1$ and $\ell = \lceil 1.9665k \rceil$. We show that $H = (k \cdot K_3) \sqcup (\ell \cdot K_2)$ is uncommon. Define $\alpha = \ell/k$ and note that $1.9665 \le \alpha \le 2$. Let

$$p := 1 - 2^{-1/(3+\alpha)}.$$

We let $W$ be the graphon $W_{\vec{z},A}$ where $\vec{z} = (1/2, 1/2)$ and $A$ is a $2 \times 2$ matrix whose diagonal entries are $p$ and off-diagonal entries are 1.

**Proposition 10.** *The graph $(2 \cdot K_3) \sqcup (3 \cdot K_2)$ is uncommon*

*Proof.* We prove that the graph $H = (2 \cdot K_3) \sqcup (3 \cdot K_2)$ is uncommon. For $z \in [0, 1/2]$ and $y \in [0,1]$, we define $W_{z,y} := W_{\vec{z},A}$ where $\vec{z} = (1-2z, z, z) \in \triangle_3$ and $A$ is the symmetric $3 \times 3$ matrix in which $A(1,2) = A(1,3) = 1$, $A(2,3) = y$ and $A(i,i) = 0$ for $1 \le i \le 3$. Setting $z = 0.28$ and $y = 0.42$ yields $h(z,y) = 0.00390226 < 2 \cdot \left(\frac{1}{2}\right)^9$, which completes the proof. □

**Proposition 11.** *$(3 \cdot P) \sqcup (2 \cdot K_2)$ is uncommon.*

*Proof.* Let $H = (3 \cdot P) \sqcup (2 \cdot K_2)$. Once again, we use the graphon $W_{z,y}$ from the previous three proofs. This time, we set $z = 0.429919$ and $y = 0.43222$. Thus, $t(H, W_{z,y}) + t(H, 1 - W_{z,y}) < 0.000121856 < 2(1/2)^{14}$ and the result follows.

□

Using the same construction above with different values of $y$ and $z$, we could get the upper bound of Theorem 5 and Theorem 6

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