Integer programs with nearly totally unimodular matrices: the cographic case

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Abstract

It is a notorious open question whether integer programs (IPs), with an integer coefficient matrix M whose subdeterminants are all bounded by a constant Δ in absolute value, can be solved in polynomial time. We answer this question in the affirmative if we further require that, by removing a constant number of rows and columns from M, one obtains a submatrix A that is the transpose of a network matrix.

Our approach focuses on the case where A arises from M after removing k rows only, where k is a constant. We achieve our result in two main steps, the first related to the theory of IPs and the second related to graph minor theory.

First, we derive a strong proximity result for the case where A is a general totally unimodular matrix: Given an optimal solution of the linear programming relaxation, an optimal solution to the IP can be obtained by finding a constant number of augmentations by circuits of A.

Second, for the case where A is transpose of a network matrix, we reformulate the problem as a maximum constrained integer potential problem on a graph G. We observe that if G is 2-connected, then it has no rooted $K_{2,t}$-minor for $t = \Omega(k\Delta)$. We leverage this to obtain a tree-decomposition of G into highly structured graphs for which we can solve the problem locally. This allows us to solve the global problem via dynamic programming.

1 Introduction

As for most computational problems that are NP-hard, the mere input size of an integer program (IP) does not seem to capture its difficulty. Instead, several works have identified additional parameters that significantly influence the complexity of solving IPs. These include the number of integer variables (Lenstra [LJ83], see also [Kan87, Dad12, RR23]), the number of inequalities for IPs in inequality form (Lenstra [LJ83]), the number of equations for IPs in equality form (Papadimitriou [Pap81],

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see also [EW19]), and features capturing the block structure of coefficient matrices (see for instance [CEH+21, CEP+21, EHK+22, BKK+24, CKL+24]).

Another parameter that has received particular interest is the *largest subdeterminant* of the coefficient matrix, which already appears in several works concerning the complexity of linear programs (LPs) and the geometry of their underlying polyhedra [Tar86, DF94, BDSE+14, EV17] as well as proximity results relating optimal solutions of IPs and their LP relaxations [CGST86, PWW20, CKPW22]. Consider an IP of the form

$$\max \left\{ p^{\intercal} x : Mx \leq b,\, x \in \mathbb{Z}^n \right\}, \tag{IP}$$

where M is an integer matrix that is *totally Δ-modular*, i.e., the determinants of square submatrices of M are all in $\{-\Delta, \dots, \Delta\}$. It is a basic fact that if M is totally unimodular ($\Delta = 1$), then the optimum value of (IP) is equal to the optimum value of its LP relaxation, implying that (IP) can be solved in polynomial time. In a seminal paper by Artmann, Weismantel & Zenklusen [AWZ17], it is shown that (IP) is still polynomial-time solvable if $\Delta = 2$, leading to the conjecture that this may hold for every constant Δ. Recently, Fiorini, Joret, Yuditsky & Weltge [FJWY22] answered this in the affirmative under the further restriction that M has only two nonzeros per row or column. In particular, they showed that in this setting, (IP) can be reduced to the stable set problem in graphs with bounded odd cycle packing number [BFMRV14, CFHW20, CFH+20].

We remark that the algorithm of [AWZ17] even applies to full column rank matrices $M \in \mathbb{Z}^{m \times n}$ for which only the $(n \times n)$-subdeterminants are required to be in $\{-\Delta, \dots, \Delta\}$ for $\Delta = 2$. Further results supporting the conjecture have been recently obtained by Nägele, Santiago & Zenklusen [NSZ22] and Nägele, Nöbel, Santiago & Zenklusen [NNSZ23] who considered the special case where all size-$(n \times n)$ subdeterminants are in $\{-\Delta, 0, \Delta\}$. Interestingly, the results of [AWZ17, NSZ22, NNSZ23] are crucially centered around a reformulation of (IP) where M becomes *totally unimodular up to removing a constant number of rows*, where the additional constraints capture a constant number of congruency constraints.

In an effort to provide more evidence for the above conjecture, we initiate the study of IPs in which M is totally Δ-modular and *nearly totally unimodular*, i.e., M becomes totally unimodular after removing a constant number of rows and columns. Note that without requirements on the subdeterminants, IPs with nearly totally unimodular coefficient matrices are still NP-hard. A famous example is the densest k-subgraph problem [BCC+10, Man17], which can be seen as an IP defined by a totally unimodular matrix with two extra rows (modeling a single equality constraint). A closely related example is the partially ordered knapsack problem [KS02], which is also strongly NP-hard. Another famous example is the exact matching (or red-blue matching) problem [Maa22, MVV87], for which no deterministic polynomial-time algorithm is known (yet).

While settling the conjecture for nearly totally unimodular coefficient matrices still seems to be a difficult undertaking, we can solve it for an important case: A celebrated result by Seymour [Sey80] states that network matrices and their transposes are the main building blocks of totally unimodular matrices. To any given (weakly) connected directed graph G and spanning tree T of G, one associates the *network matrix* $A \in \{0, \pm 1\}^{E(T) \times E(G-T)}$ such that $A_{e,(v,w)}$ is equal to 1 if e is traversed in forward direction on the unique v-w-path in T, is equal to -1 if it is traversed in backward direction, and is equal to 0 otherwise. Our main result is the following.

Theorem 1. *There is a strongly polynomial-time algorithm for solving the integer program* (IP) *for the case where M is totally Δ-modular for some constant Δ and becomes the transpose of a network matrix after removing a constant number of rows and columns.*

We achieve our result in two main steps, one related to the theory of integer programming and one related to graph minor theory. For the first step, we derive a new proximity result on distances between optimal solutions of IPs and their LP relaxations. A classic result of this type was established by Cook, Gerards, Schrijver, & Tardos [CGST86] who showed that if M is totally Δ-modular, (IP) is feasible, and x^* is an optimal solution of the LP relaxation, then there exists an optimal solution z^* of (IP) with $\|x^* - z^*\|_\infty \leq n\Delta$. It is still open whether this bound can be replaced with a function in Δ only, see [CKPW22].

A convenient consequence of this result is that, given x^*, one can efficiently enumerate the possible values of z^* for a constant number of variables. In particular, if we are given the integer program (IP) with a totally Δ-modular coefficient matrix M that becomes totally unimodular after removing k rows and ℓ columns, we may simply guess the values of the variables corresponding to the ℓ columns and solve a smaller IP for each guess.

Thus, we may assume that M (is totally Δ-modular and) is of the form $M = \left[\begin{smallmatrix} A \\ W \end{smallmatrix}\right]$, where A is totally unimodular and W is an integer matrix with only k rows. For this class of IPs, we derive a considerably strengthened proximity result: Given an optimal solution x^* of the corresponding LP relaxation, there is an optimal solution z^* of (IP) where $\|x^* - z^*\|_\infty \leq f(k, \Delta)$ for some function f that depends only on k and Δ, again provided that (IP) is feasible. In fact, by bringing (IP) into equality form, we show that x^* can be rounded to a closeby integer point from which z^* can be reached by adding a number of conformal *circuits* of $\begin{bmatrix} A & \mathbf{I} \end{bmatrix}$ that can be bounded in terms of k and Δ only. Moreover, we observe that the fact that M is totally Δ-modular implies that every circuit c satisfies $\|Wc\|_\infty \leq \Delta$.

While these findings are valid for all totally unimodular matrices A, we will see that they can be crucially exploited for the case where A is the transpose of a network matrix, which we refer to as the *cographic case*. For these instances, it is convenient to reformulate the original problem (IP) as a particular instance of a *maximum constrained integer potential problem*

$$\max\left\{p^\intercal y : \ell(v,w) \leq y(v) - y(w) \leq u(v,w) \text{ for all } (v,w) \in E(G),\, Wy \leq d,\, y \in \mathbb{Z}^{V(G)}\right\}, \quad \text{(MCIPP)}$$

where G is a directed graph, $p \in \mathbb{Z}^{V(G)}$, $\ell, u \in \mathbb{Z}^{E(G)}$, $W \in \mathbb{Z}^{[k]\times V(G)}$ and $d \in \mathbb{Z}^k$, and moreover each row of $p^\intercal$ or W sums up to zero. Notice that the first constraints are still given by a totally unimodular matrix, and hence we may regard $Wy \leq d$ as extra (or complicating) constraints. With this reformulation, the circuits of $\begin{bmatrix} A & \mathbf{I} \end{bmatrix}$ turn into vertex subsets $S \subseteq V(G)$ with the property that both induced subgraphs $G[S]$ and $G[\overline{S}]$ are (weakly) connected, where $\overline{S} := V(G) \setminus S$. We call such sets *doubly connected sets* or *docsets*. Using this notion, our previous findings translate to two strong properties of the instances of (MCIPP) we have to solve: First, every feasible instance has an optimal solution that is the sum of at most $f(k, \Delta)$ incidence vectors $\chi^S \in \{0,1\}^{V(G)}$, where S is a docset. Second, every docset S satisfies $\|W\chi^S\|_\infty \leq \Delta$.

Referring to the vertices whose variables appear with a nonzero coefficient in at least one of the extra constraints as *roots*, the second property above implies that roots cannot be arbitrarily distributed in the input graph. Roughly speaking, by carefully exploiting the structure of the instance, we will be able to guess $y(v)$ for each root v. Note that once all of these variables are fixed, the resulting IP becomes easy since its constraint matrix is totally unimodular. In fact, the guessing cannot be done for the whole graph at once and we will have to do it locally, and then combine the local optimal solutions via dynamic programming.

Our structural insights are based on the observation that our input graphs do not contain a *rooted $K_{2,t}$-minor*, where $t = 4k\Delta + 1$, provided that they are 2-connected. Here, the minors of a rooted graph (graph with a distinguished set of vertices called roots) are defined similarly as for usual graphs, with two differences: whenever some edge e is contracted we declare the resulting vertex as a root if and only if at least one of its ends is a root, and we have the possibility to remove a vertex from the set of roots. A rooted $K_{2,t}$ is said to be *properly rooted* if each one of the t vertices in the "large" side is a root. For the sake of simplicity, we say that a rooted graph contains a *rooted $K_{2,t}$-minor* if it has a rooted minor that is a properly rooted $K_{2,t}$, see Figure 1.

Our main structural result is a decomposition theorem for rooted graphs without rooted $K_{2,t}$-minor, see Theorem 2 below. It relies partly on several works about the structure of graphs excluding a minor, extending the original results of Robertson & Seymour within the graph minors project, more specifically on works by Diestel, Kawarabayashi, Müller & Wollan [DiKMW12], Kawarabayashi, Thomas

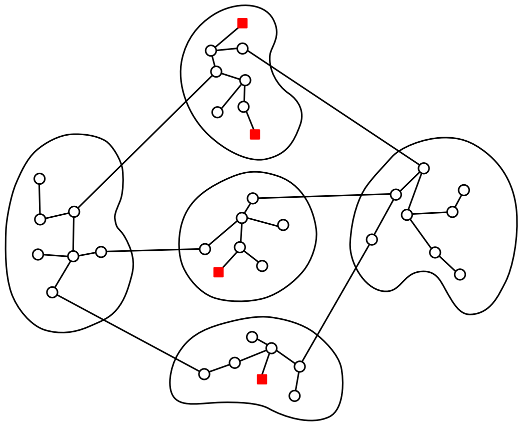

Figure 1: Subgraph containing a rooted $K_{2,3}$-minor. Roots are indicated with the red squares. Contracting all the edges in each of the five *branch sets* produces a properly rooted $K_{2,3}$.

& Wollan [KTW20], and Thilikos & Wiederrecht [TW22]. Furthermore, we use results of Böhme & Mohar [BM02] and Böhme, Kawarabayashi, Maharry & Mohar [BKMM08], to control the distribution of the roots in surface-embedded rooted graphs without rooted $K_{2,t}$-minors.

Our decomposition theorem is formulated in terms of a tree-decomposition of graph G. Recall that a *tree-decomposition* is a pair $(T, \mathcal{B})$ where T is a rooted tree (tree with a unique *root node*) and $\mathcal{B} = \{B_u : u \in V(T)\}$ is a collection of vertex subsets of G, called *bags*, such that for every vertex v of G the set of bags containing v induces a non-empty subtree of T, and for every edge e of G there is a bag that contains both ends of e. We define the *weak torso* of a bag B_u as the graph obtained from the induced subgraph $G[B_u]$ by adding a clique on $B_u \cap B_{u'}$ for each node $u' \in V(T)$ that is a child of u. Having stated these definitions, we are ready to state the (simplified version of our) decomposition theorem. See Figure 2 for an illustration.

Theorem 2 (simplified version)**.** *For every $t \in \mathbb{Z}_{\geq 1}$ there exists a constant $\ell = \ell(t)$ such that every 3-connected rooted graph G without a rooted $K_{2,t}$-minor admits a tree-decomposition $(T, \mathcal{B})$, where $\mathcal{B} = \{B_u : u \in V(T)\}$, with the following properties:*

(i) the bags B_u and $B_{u'}$ of two adjacent nodes $u, u' \in V(T)$ have at most ℓ vertices in common, and

(ii) for every node $u \in V(T)$, all but at most ℓ children $u' \in V(T)$ of u are leaves and the roots contained in the corresponding bags $B_{u'}$ are all contained in bag B_u, and

(iii) every node $u \in V(T)$ satisfies one of the following:

(a) bag B_u has at most ℓ vertices, or

(b) u is a leaf and B_u has at most ℓ roots, all contained in the bag of the parent of u, or

(c) after removing at most ℓ vertices, the weak torso of B_u becomes a 3-connected rooted graph that does not contain a rooted $K_{2,t}$-minor and has an embedding in a surface of Euler genus at most ℓ such that every face is bounded by a cycle, and all its roots can be covered by at most ℓ facial cycles.

As we show, there is a polynomial-time algorithm that finds the tree-decomposition of Theorem 2 together with a polynomial-size collection $\mathcal{X}_u$ for each node $u \in V(T)$, containing all the possible intersections of a docset of G with the roots contained in bag B_u. This yields an efficient dynamic programming algorithm to solve the instances of (MCIPP) we are interested in, which proves Theorem 1.

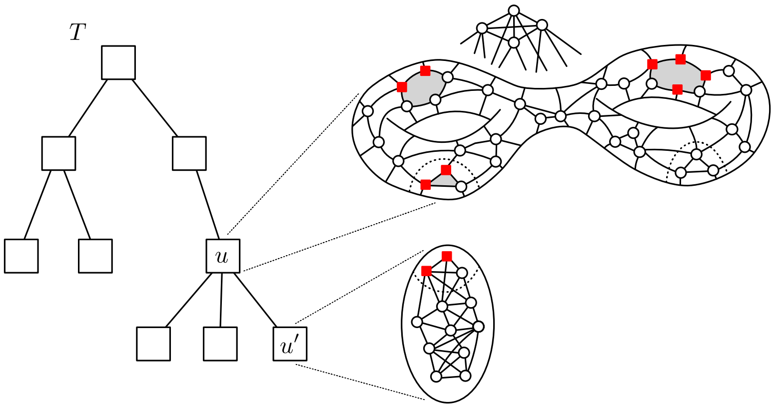

Figure 2: Illustrating the decomposition of Theorem 2. The decomposition tree T is shown on the left. The weak torsos of the two bags B_u and $B_{u'}$ are shown on the right. The top one satisfies (iii).(c), and the bottom one (iii).(b).

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