Switching methods for the construction of cospectral graphs*

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1 Introduction

An important problem in algebraic graph theory is to decide whether a graph is determined by the spectrum of its adjacency matrix (see the surveys [10, 11]). In 2003, van Dam and Haemers [10] conjectured that almost all graphs are uniquely determined by their spectrum. While the conjecture is still open, Brouwer and Spence [8] provided computational evidence by enumerating all graphs with up to 12 vertices and observing a decline in the fraction of *cospectral mates* (non-isomorphic graphs with the same spectrum) between 10 and 12 vertices. Recent work by Koval and Kwan [17] showed that an exponential number of graphs is determined by its spectrum. On the other side, Haemers and Spence [14] established an asymptotic lower bound for the number of cospectral mates. Their key ingredient is the notion of switching.

A *switching method* is an operation on a graph that results in a graph with the same spectrum. For such a method to work, the graph needs a special structure, called a *switching set*. This set of vertices makes it possible to swap some of the edges while preserving the spectrum of the adjacency matrix. While Godsil-McKay (GM) switching [13] is the oldest and most fruitful switching method in the literature (see e.g. [2, 3, 4]), new switching methods have recently been presented in the literature, most notably Wang-Qiu-Hu (WQH) switching [20] and Abiad-Haemers (AH) switching [4]. The latter captures all level 2 switching methods, and is motivated by the results of Wang and Xu [21], who suggested that almost all $\mathbb{R}$-*cospectral graphs* (cospectral graphs with cospectral complements) can be constructed using regular orthogonal matrices of level 2.

This work bridges a gap in the existing literature concerning the recently introduced switching methods of level 2. In particular, we present a combinatorial description of AH-switching that is more accessible than the algebraic description provided by Abiad and Haemers in [5]. We do this for switching sets of sizes 6, 8 and 10. Moreover, we show that the asymptotic lower bound on cospectral mates derived by Haemers and Spence [14] is tight for GM-switching. We also obtain analogous upper and lower bounds on the number of cospectral mates obtained via WQH-switching.

2 Preliminaries

In this work, graphs are considered to be simple and loopless. The *(adjacency) spectrum* of a graph is the multiset of eigenvalues of its adjacency matrix. Graphs are *cospectral* if they have the same spectrum. Two graphs are said to be *cospectral mates* if they are cospectral and non-isomorphic. Let I

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denote the identity matrix and J the all-one matrix. Two graphs with adjacency matrices A and A' are called $\mathbb{R}$-*cospectral* if $A+rJ$ and $A'+rJ$ are cospectral for every $r \in \mathbb{R}$. An orthogonal matrix is *regular* if it has a constant row sum. Johnson and Newman [16] showed that two graphs are $\mathbb{R}$-cospectral if and only if their adjacency matrices are conjugated with a regular orthogonal matrix.

The *level* of a matrix is the smallest positive integer ℓ such that ℓ times the matrix is an integral matrix, or ∞ if it has irrational entries. A matrix is *decomposable* if it can be written as a non-trivial block-diagonal matrix after a certain permutation of the rows and columns. Otherwise, it is *indecomposable*.

3 Switching methods to construct cospectral graphs

The construction of cospectral graphs has multiple purposes: to disprove the conjecture stating that almost all graphs can be characterized by their spectrum for certain graph classes (see e.g. [3, 12]), to show which properties of a graph cannot be deduced from the spectrum (see e.g. [1, 7, 18]), or to construct new strongly regular and distance-regular graphs (see e.g. [6, 19]), among others. In what follows, we provide an overview of the existing switching methods, and present some new results concerning AH-switching.

3.1 Godsil-McKay switching

The following method for finding cospectral graphs was introduced by Godsil and McKay [13] in 1982.

Theorem 1 (GM-switching [13])**.** *Let Γ be a graph and let $\{C_1, \dots, C_t, D\}$ be a partition of its vertices such that, for all $i, j \in \{1, \dots, t\}$:*

(i) Every vertex in C_i has the same number of neighbours in C_j.

(ii) Every vertex in D has 0, $\frac{1}{2}|C_i|$ or $|C_i|$ neighbours in C_i.

For all $i \in \{1, \dots, t\}$ and every $v \in D$ that has exactly $\frac{1}{2}|C_i|$ neighbours in C_i, swap the adjacencies between v and C_i. The resulting graph is $\mathbb{R}$-cospectral with Γ.

The GM-switching operation corresponds to a conjugation of the adjacency matrix with the orthogonal matrix $\text{diag}(R_1, \dots, R_t, I)$, where R_i equals the $|C_i| \times |C_i|$ matrix $\frac{2}{|C_i|}J - I$. Note that any C_i of order 2 only gives a permutation matrix and is therefore trivial. The simplest nontrivial case has one switching block of size four. This case has actually been the most fruitful in the literature, see e.g. [2, 3, 4]. Larger switching sets give more conditions on the graph, which intuitively explains the relative effectiveness of small switching sets. In Section 4, we give an asymptotic formula for the number of graphs with a switching set of size four. Note that the level of the corresponding matrix is 2 in that case, and the lowest common multiple of $\frac{1}{2}|C_i|$, $1 \le i \le t$, in general.

3.2 Wang-Qui-Hu switching

In 2019, Wang, Qiu and Hu [20] presented another switching method, which corresponds to a conjugation of the adjacency matrix with the orthogonal matrix $\text{diag}(R_1, \dots, R_t, I)$, where each R_i is of the form

$$R_i = \begin{pmatrix} I - \frac{2}{|C_i|}J & \frac{2}{|C_i|}J \\ \frac{2}{|C_i|}J & I - \frac{2}{|C_i|}J \end{pmatrix}.$$

As illustrated in [3, 12, 15], WQH-switching is also a powerful tool for constructing cospectral graphs in cases where GM-switching fails. In combinatorial terms, the method can be described as follows.

Theorem 2 (WQH-switching [20])**.** *Let Γ be a graph and let $\{C_1^{(1)}, C_1^{(2)}, \dots, C_t^{(1)}, C_t^{(2)}, D\}$ be a partition of its vertices such that, for all $i, j \in \{1, \dots, t\}$:*

(i) $|C_i^{(1)}| = |C_i^{(2)}|$.

(ii) The number $\begin{cases} |N(v)\cap C_j^{(1)}| - |N(v)\cap C_j^{(2)}| & \text{if } v\in C_i^{(1)} \\ |N(v)\cap C_j^{(2)}| - |N(v)\cap C_j^{(1)}| & \text{if } v\in C_i^{(2)} \end{cases}$ *is the same for every* $v \in C_i^{(1)} \cup C_i^{(2)}$.

(iii) Every vertex in D *has either:*

(a) $|C_i^{(1)}|$ *neighbours in* $C_i^{(1)}$ *and* 0 *neighbours in* $C_i^{(2)}$,

(b) 0 *neighbours in* $C_i^{(1)}$ *and* $|C_i^{(2)}|$ *neighbours in* $C_i^{(2)}$,

(c) the same number of neighbours in $C_i^{(1)}$ *as in* $C_i^{(2)}$.

For all $i \in \{1,\dots,t\}$ *and every* $v \in D$ *for which (a) or (b) holds, swap the adjacencies between* v *and* $C_i^{(1)} \cup C_i^{(2)}$. *The resulting graph is* $\mathbb{R}$*-cospectral with* Γ.

If $t = 1$ and $|C_1^{(1)}| = |C_2^{(2)}| = 2$, then WQH-switching is equivalent to GM-switching on $C_1^{(1)} \cup C_1^{(2)}$. But in general, they are different operations. Note that the level of the corresponding matrix is equal to the lowest common multiple of $\frac{1}{2}|C_i|$, $1 \le i \le t$, just like for GM-switching.

3.3 Abiad-Haemers switching

In 2012, Abiad and Haemers [5] considered switching methods that correspond to a conjugation of the adjacency matrix with a regular orthogonal matrix of level 2. In particular, these methods can be used to construct $\mathbb{R}$-cospectral graphs (see the characterization of $\mathbb{R}$-cospectral graphs by Johnson and Newman [16]). Their starting point is the following classification of indecomposable regular orthogonal matrices of level 2, which follows from the classification of weighing matrices of weight 4 by Chan, Rodger and Seberry [9] and has been restated by Wang and Xu [21] in the form below. Note that any regular orthogonal matrix has row sum 1 or -1, but without loss of generality, we may assume the row sum to be 1.

Theorem 3 ([9])**.** *Up to a permutation of the rows and columns, an indecomposable regular orthogonal matrix of level* 2 *and row sum* 1 *is one of the following:*

$$(i)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},\quad (ii)\ \frac{1}{2}\begin{bmatrix} J & O & \cdots & \cdots & O & Y \\ Y & J & O & \cdots & \cdots & O \\ O & Y & J & O & \cdots & O \\ & \ddots & \ddots & \ddots & \ddots & \\ O & \cdots & O & Y & J & O \\ O & \cdots & \cdots & O & Y & J \end{bmatrix},$$

$$(iii)\ \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix},\quad (iv)\ \frac{1}{2}\begin{bmatrix} -I & I & I & I \\ I & -Z & I & Z \\ I & Z & -Z & I \\ I & I & Z & -Z \end{bmatrix},$$

where I, J, O, $Y = 2I - J$ *and* $Z = J - I$, *are square matrices of order* 2.

The matrix in Theorem 3(i) corresponds to GM-switching. The one in Theorem 3(ii) is an infinite family of matrices of even order, starting from order 6. In the following, we focus on this infinite family. The remaining matrices in Theorem 3(iii)-(iv) were studied by Abiad and Haemers in [5, Section 5 and Section 6].

It was already noticed by Abiad and Haemers [5] that sometimes, the six vertex AH-switching can be obtained by GM-switching twice. We make this notion concrete in the following new definition.

Definition 4. *Let Q be a regular orthogonal matrix of level 2 and let A be an adjacency matrix with the property that $Q^T AQ$ is again an adjacency matrix. Then A is called* reducible *with respect to Q if there exist regular orthogonal matrices Q_1 and Q_2 of level 2 whose largest indecomposable block is smaller than that of Q, such that $Q = Q_1 Q_2$ and $Q_1^T B Q_1$ is also an adjacency matrix. Otherwise, it is called* irreducible*.*

In what follows, we describe only the irreducible adjacency matrices for the AH-switching set, since the reducible ones can be obtained by repeated GM- and AH-switching on smaller sets.

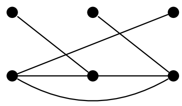
3.3.1 Six vertex switching

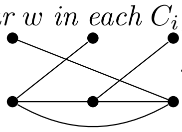
We present a combinatorial description of the switching on 6 vertices that was established by Abiad and Haemers [5, Section 4]. Recall that this switching corresponds to a conjugation of the adjacency matrix with the matrix in Theorem 3(ii) of order 6.

Theorem 5 (AH6-switching)**.** *Let Γ be a graph and let $\{C_1, C_2, C_3, D\}$ be a partition of its vertices such that:*

(i) $|C_1| = |C_2| = |C_3| = 2$.

(ii) Every vertex in D has the same number of neighbours in C_1, C_2 and C_3 modulo 2.

(iii) The induced subgraph on $C_1 \cup C_2 \cup C_3$ is  *(in that order, from left to right).*

Let π be the permutation on $C_1 \cup C_2 \cup C_3$ that shifts the vertices cyclically to the right. For every $v \in D$ that has exactly one neighbour w in each C_i, replace each edge $\{v, w\}$ by $\{v, \pi(w)\}$. Replace the induced subgraph on $C_1 \cup C_2 \cup C_3$ by *. The resulting graph is $\mathbb{R}$-cospectral with Γ.*

Among the seven possible adjacency matrices for an AH-switching set of size 6, obtained by Abiad and Haemers in [5, Lemma 6], only two are irreducible. However, they are actually equivalent, and correspond to the induced subgraph in the statement of Theorem 5. In other words:

Theorem 6. *AH6-switching is the only switching that corresponds to a regular orthogonal matrix of level 2 with one indecomposable block of size 6 and that cannot be obtained by repeated GM-switching.*

3.3.2 Eight vertex switching

Surprisingly, all matrices that describe an AH-switching set of order 8 (corresponding to the matrix of order 8 in the infinite family of Theorem 3(ii)) are reducible:

Theorem 7. *Every switching that corresponds to a conjugation with the matrix* $\frac{1}{2}\begin{bmatrix} J & O & O & Y \\ Y & J & O & O \\ O & Y & J & O \\ O & O & Y & J \end{bmatrix}$ *can be obtained by repeated GM- and AH6-switching.*

3.3.3 Ten vertex switching

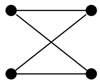
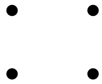
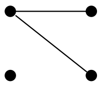
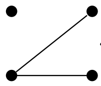
In contrast with the eight vertex case, there are $3 \cdot 2^{10} = 3072$ possibilities for an (irreducible) AH-switching set of size 10.

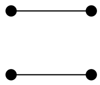
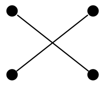
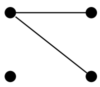
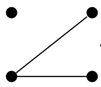
Theorem 8 (AH10-switching)**.** *Let Γ be a graph and let $\{C_1, C_2, C_3, C_4, C_5, D\}$ be a partition of its vertices such that:*

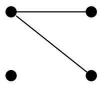
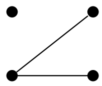
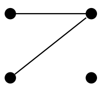
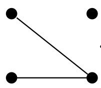
(i) $|C_1| = |C_2| = |C_3| = |C_4| = |C_5| = 2$.

(ii) Every vertex in D *has the same number of neighbours in* C_1, C_2, C_3, C_4 *and* C_5 *modulo* 2.

(iii) One of the following holds (vertices are ordered, from left to right):

(a) For every $i \in \mathbb{Z}/5\mathbb{Z}$, *the induced subgraph on* $C_i \cup C_{i+1}$ *is either*  *or*  *and the induced subgraph on* $C_i \cup C_{i+2}$ *is either*  *or* .

(b) For every $i \in \mathbb{Z}/5\mathbb{Z}$, *the induced subgraph on* $C_i \cup C_{i+1}$ *is either*  *or*  *and the induced subgraph on* $C_i \cup C_{i+2}$ *is either*  *or* .

(c) For every $i \in \mathbb{Z}/5\mathbb{Z}$, *the induced subgraph on* $C_i \cup C_{i+1}$ *is either*  *or*  *and the induced subgraph on* $C_i \cup C_{i+2}$ *is either*  *or* .

Let π *be the permutation on* $C_1 \cup \cdots \cup C_5$ *that shifts the vertices cyclically to the right. For every* $v \notin C$ *that has exactly one neighbour* w *in each* C_i, *replace each edge* $\{v, w\}$ *by* $\{v, \pi(w)\}$. *Replace the induced subgraph on* $C_1 \cup \cdots \cup C_5$ *by the unique graph such that, according to the cases above:*

(a) For every $i \in \mathbb{Z}/5\mathbb{Z}$, *the induced subgraph on* $C_i \cup C_{i+1}$ *remains invariant, and the new induced subgraph on* $C_i \cup C_{i+2}$ *is the former induced subgraph on* $C_i \cup C_{i+3}$.

(b) For every $i \in \mathbb{Z}/5\mathbb{Z}$, *the new induced subgraph on* $C_i \cup C_{i+1}$ *is the former induced subgraph on* $C_{i+1} \cup C_{i+2}$ *and the new induced subgraph on* $C_i \cup C_{i+2}$ *is the former induced subgraph on* $C_i \cup C_{i+3}$.

(c) For every $i \in \mathbb{Z}/5\mathbb{Z}$, *the new induced subgraph on* $C_i \cup C_{i+1}$ *is the former induced subgraph on* $C_{i-1} \cup C_{i+1}$ *and the new induced subgraph on* $C_i \cup C_{i+2}$ *is the former induced subgraph on* $C_{i+1} \cup C_{i+2}$.

The resulting graph is $\mathbb{R}$*-cospectral with* Γ.

Similar to the six vertex case, we have the following result:

Theorem 9. *AH10-switching is the only switching that corresponds to a regular orthogonal matrix of level* 2 *with one indecomposable block of size* 10 *and that cannot be obtained by repeated GM- and AH6-switching.*

4 Asymptotic bounds

Let g_n denote the number of graphs on n (unlabelled) vertices. In 2005, Haemers and Spence [14] established a lower bound on the number of graphs on n vertices that have a cospectral mate.

Theorem 10 ([14, Theorem 3])**.** *There are at least* $n^3 g_{n-1}(\frac{1}{24} - o(1))$ *graphs on* n *vertices with a cospectral mate.*

This bound was derived by counting the number of cospectral mates by GM-switching with respect to a switching set of size 4. Therefore, it is also a lower bound on the number of graphs which have a cospectral mate via GM-switching. From their proof, we can also deduce a matching upper bound.

Theorem 11. *There are* $n^3 g_{n-1}\left(\frac{1}{24} + o(1)\right)$ *non-isomorphic graphs on* n *vertices with a GM-switching set of size* 4.

Analogous bounds can be obtained for WQH-switching. Intuitively, there are less graphs with a WQH-switching set of size 6, because they require more conditions than a switching set of size 4.

Theorem 12. *There are between*

$$n^4 g_{n-2}(\frac{1}{72} - o(1)) \qquad \text{and} \qquad n^4 g_{n-2}(\frac{11}{8})^{n-6}(2^9 + o(1))$$

non-isomorphic graphs on n vertices with a cospectral mate that can be obtained via WQH-switching on 6 vertices.

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